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THE INTERRELATION OF THE EYES OF PALAEMONETES AS CONCERNS RETINAL PIGMENT MIGRATION

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Whether "sympathetic" migration of the retinal pigment may occur in crustaceans when one eye is in darkness and the other illuminated has been in doubt. Parker (1897) working with Palaemonetes used a balsam-lampblack coating for one eye of a light-adapted animal, allowing the other eye to be freely exposed to the light. In another experiment he enclosed an animal in a light-tight box, with one eyestalk protruding through a hole into the light. In both cases he found that the eyes exposed to the light presented the condition normal for the light, while those kept in the dark showed an incomplete approach to the normal dark condition. He concluded that leakage of light through the optic stalk into the eye explained the incompleteness of the dark adaptation, and that the retinal pigment migrations of the two eyes of Palaemonetes were independent of one another and of central nervous organs.

Bennitt (1924) found with Cambarus that in a light-adapted animal with one eye blackened by a celloidin-lampblack mixture, exposure of the uncovered eye to light prevented complete dark adaptation in the covered eye. He further concluded that the photo-mechanical changes in the amphipod eye were partly controlled by the central nervous system.

It was desired to test this question of the "sympathetic" interrelation of the eyes of Palaemonetes with a more rigorous technique. Coats of celloidin in alcohol-ether solution were frequently found to destroy the cone layer of the eye, and such solvents might well influence the pigment migrations taking place close by. Eyes covered with a cast of plaster of Paris enclosing the whole eye and eyestalk produced no destructive changes, and an equal mixture of lampblack and plaster of Paris was used in all the following experiments as a safe opaque covering. The eyes were fixed in water at 80°C., and cut in paraffine.

Using light-adapted animals, afterward kept in the light for over an hour, both eyes were coated with the opaque mixture to see whether

complete dark adaptation could be obtained. Pigment is present in two principal places in the decapod eye: *distal pigment* which moves distally and proximally on the cones, and *proximal pigment* which may move within the proximal retinular cells from a position surrounding the rhabdomes (light-adapted) inward through the so-called refractive layer of the retina to a position completely below the basement membrane (dark-adapted). It is with this proximal pigment alone that all the following experiments have to do, and in the following tabulations four arbitrary classes have been selected to describe the location of the outer edge of this pigment. The normal dark position is somewhat below the basement membrane.

SERIES 1

BOTH EYES COVERED WITH PLASTER OF PARIS-LAMPBLACK MIXTURE

1.	Pigment below the basement membrane	13 eyes
2.	Pigment above basement membrane, but below refractive layer of retina	4 eyes
3.	Pigment in the refractive layer	7 eyes
4.	Pigment just above refractive layer	2 eyes

Total number of animals, 13.
Total number of eyes, 26.

Thus in half the cases the pigment was in the position characteristic of *complete* dark adaptation. That this extreme condition can be obtained greatly discounts the irregularity of technique which probably explains the remaining cases.

Again using light-adapted animals, *one* eye was covered and the animals kept in the light for an hour or more.

SERIES 2

ONE EYE COVERED WITH PLASTER OF PARIS-LAMPBLACK MIXTURE

1.	Pigment below the basement membrane	7 eyes
2.	Pigment above basement membrane, but below refractive layer of retina	8 eyes
3.	Pigment in the refractive layer	1 eye
4.	Pigment just above refractive layer	none

Total number of animals, 16.
Total number of dark eyes, 16.

Considering together classes (1) and (2) of both series of experiments, it is evident that there is no significant difference between the two series. It is, therefore, made very probable that there is no "sympathetic" nervous interrelation of the eyes of Palaemonetes and that, as Parker concluded, the eyes are relatively local and autonomous centers as concerns their photo-mechanical changes. A further fragment of evidence for this view is that in a few eyes where the plaster of Paris cast was incomplete there are local areas where the retina pigment cells are in a normal light-adapted condition, while all around with a rather sharp zone of transition the retina is dark-adapted.

I wish to thank Prof. G. H. Parker for suggesting this work.

Summary.—1. Celloidin-alcohol-ether mixtures applied externally to the eyes of Palaemonetes tend to destroy them, but plaster of Paris-lampblack mixtures are without destructive effect.

2. If both eyes of a previously light-adapted animal are blackened, complete dark adaptation may occur. If only one eye of a similar animal is blackened, complete dark adaptation of the covered eye may also result.

3. It is probable that in Palaemonetes an illuminated eye is without "sympathetic" effect by way of the central nervous system on the photo-mechanical changes of the covered eye.

Bennitt, R., 1924, "The Migration of the Retinal Pigment in Crustaceans," *J. Exp. Zoöl.*, **40**, 381–435.

Parker, G. H., 1897, "Photomechanical Changes in the Retinal Pigment Cells of Palaemonetes, and Their Relation to the Central Nervous System," *Bull. Mus. Comp. Zoöl.*, **30**, 273–300.

THE BLOOD-VESSELS OF ANNELIDS

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The following is a preliminary report on some observations on the distribution, histology and physiology of the annelid blood-vessels. A comparison is made between these and the vertebrate capillaries. The worm employed in this study was Nereis virens Sars.

The vascular system of Nereis consists of a contractile dorsal vessel and a non-contractile ventral vessel connected one with the other in each segment of the worm by contractile lateral branches. These branches in the anterior region are enlarged to form the so-called "hearts." The histological structure of the blood-vessels was studied by the examination of sections and whole-mounts of the different vessels stained with methylene blue. This stain was used as an "intra vitam" stain, and observations thus made were corroborated by the study of sections stained in Ehrlich's hematoxyline-eosin. Four distinct layers are found in the blood vessels of Nereis: (1) an innermost continuous endothelium; (2) outside of this a continuous, structureless, homogeneous membrane, the "intima" or "cuticula;" (3) covering this membrane an "adventitia" or layer varying in structure from isolated stellate cells to almost typical smooth-muscle cells; (4) outside of all the layers on the dorsal vessel there is found a discontinuous peritoneal layer.

The vessels have the following composition. The capillaries consist of endothelium only. In larger vessels in addition to an endothelium

a "cuticula" is present as well as a third layer. This layer on the smaller vessels consists of isolated stellate cells whose branches encircle the tube. These so-called "Muskelzellen" (Retzius, 1891, 1905) are represented in the larger vessels of this type by spindle-shaped cells, circularly disposed, the free ends of which show slight branching. In still larger vessels the adventitious cells are more abundant. On the ventral surface of the dorsal blood-vessel masses of peritoneal cells can be seen. The ventral vessel has no typical "Muskelzellen." Longitudinally disposed smooth-muscle cells are found in its anterior third. No endothelium could be demonstrated in the ventral vessel.

Histological comparison between the annelid blood vessels and the vertebrate capillaries showed that both types of vessels are somewhat similar. The vertebrate capillaries consist of an inner nucleated endothelium (Krogh, 1922). Outside of this are found cells which have been called "Rouget" cells (Rouget, 1873, 1874, 1879; Vimtrup, 1922*a*, 1922*b*; Krogh, 1922; Clark and Clark, 1925*a*). These cells are comparable to the "Muskelzellen" found on the blood-vessels of worms.

In the study of the physiology of contraction in the blood-vessels of Nereis virens the dorsal vessel and the lateral hearts were employed. These vessels were studied in the intact animal and in pieces of the worm from which the central nervous system had been removed. In the latter cases the vessel to be studied was cut away along with the surrounding muscles. This tissue was then pinned out on a block of paraffin and the vessel exposed. The vessel could then be studied under the binocular microscope. In the investigation of the contractile elements concerned in the normal contraction wave it was found necessary to stain the vessels with methylene blue. The methylene blue was used as an "intra vitam" stain. This stained the "Muskelzellen" and their activities could thus be studied. The stained vessels were normal in all their responses. The action of drugs on the intact animal and also on the isolated vessels was investigated. The following results were obtained from these studies.

1. The normal contraction wave of the contractile vessels of Nereis virens is independent of any central nervous control.

2. Two types of contraction can be observed in the blood-vessels of Nereis virens. There is a normal contraction wave which is peristaltic in character. The other type of contraction is local in character and is produced only by direct stimulation. These two types of contraction seem to depend upon two different elements. The normal contraction wave is due to the independent active contraction of the endothelium. There is good evidence to show that local constriction depends upon the contractility of the "Muskelzellen." The "Muskelzellen" appear to be capable of active contraction only upon direct stimulation, whether electrical, mechanical or chemical. Cutting of any contractile blood-vessel

causes the free end to constrict. This constriction is dependent, apparently, upon the active contraction of the "Muskelzellen." Thus they prevent excessive bleeding following the cutting of such a vessel.

3. Strychnine acts upon the blood-vessel directly, bringing about a series of local constrictions. These constrictions are non-transmissible. Nicotine relaxes the contractile vessels. These two drugs appear to be antagonistic, in that, if a nicotinized worm is immersed in a solution of strychnine, normal contraction waves appear.

4. A comparative study of the contraction of the vertebrate capillaries and the annelid blood-vessels reveals the fact that the normal contraction in both types of vessels depends upon the independent contraction of the endothelium (Krogh, 1922; Vimtrup, 1922*a*, 1922*b*; Clark and Clark, 1925*b*). It seems very probable that the "Rouget" cells of the vertebrate capillary are functionally comparable to the "Muskelzellen" of the annelid blood vessel and that these cells react to direct stimulation only.

From the above results one may conclude that not only histologically but physiologically the vertebrate capillaries and the contractile blood-vessels of Nereis virens are closely comparable. The phylogenetic significance of this similarity is obvious.

A detailed discussion of these results will appear shortly.

Clark, E. R., and Clark, E. L., 1925*a*, A. "The Development of Adventitial ('Rouget') Cells on the Capillaries of Amphibian Larvae," *Amer. J. Anat.*, **35**, 239–264.

Clark, E. R., and Clark, E. L., 1925*b*, B. "The Relation of 'Rouget' Cells to Capillary Contractility," *Ibid.*, **35**, 265–282.

Krogh, A., 1922, *The Anatomy and Physiology of Capillaries*, New Haven, 276 pp.

Retzius, G., 1891, "Ueber Nervendigungen an der Parapodienborsten und ueber die Muskelzellen der Gefässewände bei den polychaeten Annulaten," *Verk. Biolog. Verein. Stockholm*, **3**, 85–89.

Retzius, G., 1905, "Ueber Muskelzellen an der Blutgefässen der Polychaeten," *Biolog. Unters, N. F.*, **12**, 75–78.

Rouget, C., 1873, "Mémoire sur le développement, la structure et les proprietés physiologiques des capillaires sanguins et lymphatiques," *Arch. physiol. norm. et path.*, **5**, 603–663.

Rouget, C., 1874, "Note sur le développement de la tunique contractile des vaisseaux," *Compt. rend. acad. sci. Paris*, **59**, 559–562.

Rouget, C., 1879, "Sur la contractilité des capillaires sanguins," *Compt. rend. acad. sci. Paris*, **88**, 916–918.

Vimtrup, B., 1922*a*, "Beiträge zur Anatomie der Capillaren. I. Ueber contractile Elements in der Gefässewand der Blutcapillaren," *Zeitschr. Anat. Entwickl.*, **65**, 150–182.

Vimtrup, B., 1922*b*, "Sur les éléments contractiles dans la paroi des capillaires sanguins," *Compt. rend. soc. biol.*, **87**, 761–764.

THE EFFECT OF A ROTATION OF THE GALAXY ON PROPER MOTIONS IN RIGHT ASCENSION AND DECLINATION

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The mean parallactic motions of stars in the equatorial zone $-2°$ to $+2°$, for different spectral types, have been studied by Miss Barney and me in a recent paper.[1] In another paper of mine,[2] a systematic difference in the drift of giant stars relative to the stars of the main sequence has been investigated in further detail. In order to get additional evidence of the mean parallactic motion of AG stars of different spectral types I utilized the proper motion by Gyllenberg.[3]

Plotting the mean proper motions in right ascension and in declination, it is evident that they cannot be represented by simple sine curves. I first attempted to ascribe the deviations from sine curves to systematic errors in the catalogue places. On account of the large size of the effect it seems hardly possible to do so. I have also tried to account for the deviations by ascribing them to the fact that parallactic motions are less in the Milky Way than near its poles. If so the F stars, which show no galactic concentration among the magnitudes here considered, should not show the deviations, or at least should do so to a lesser degree. This was not found to be the case. The phenomenon may be due to a rotation of the stellar system as suggested by Lindblad.[4]

Recently Oort[5] has investigated the observed term in the radial velocities of distant objects of the form $a \cdot \sin 2l + b \cdot \cos 2l$, under the assumption that there is a rotation about the center of the star system and that the rotational velocity depends only on the distance from this center.

The proper motions of the Gyllenberg stars seems to show a similar term having the argument 2α. The expressions for the velocity components are, therefore, assumed to be in rectangular galactic coördinates:

$$\begin{aligned} \dot{\xi} &= \omega_1 \cos b \sin l - \omega_2 \cos b \cos l \\ \eta &= \omega_3 \cos b \cos l + \omega_4 \cos b \sin l \\ \zeta &= 0 \end{aligned}$$

where b and l are the galactic latitude and longitude, respectively.

If we assume the motions in space to be parallel and that the amount of velocity depends only on the distance from the center of rotation we have

$$\begin{aligned} \omega_1 &= P \cos 2l_0 - Q \\ \omega_2 &= P \sin 2l_0 \\ \omega_3 &= P \cos 2l_0 + Q \\ \omega_4 &= P \sin 2l_0, \end{aligned}$$

where P and Q are the same as A and B in Oort's paper (loc. cit.).

The galactic pole is adopted to be at $\alpha = 190°$, $\delta = +28°$, and for the transformation to equatorial components we have:

$$\begin{aligned} \dot{X} &= 0.17\,\dot{\xi} + 0.46\,\dot{\eta} - 0.87\,\dot{\zeta} \\ \dot{Y} &= -0.98\,\dot{\xi} + 0.08\,\dot{\eta} - 0.15\,\dot{\zeta} \\ \dot{Z} &= \phantom{-0.98\,\dot{\xi}} + 0.88\,\dot{\eta} + 0.47\,\dot{\zeta} \end{aligned}$$

$$\begin{aligned} \text{Also } \mu_\alpha \cos\delta &= -\dot{X}\sin\alpha + \dot{Y}\cos\alpha \\ \mu_\delta &= -\dot{X}\cos\alpha\sin\delta - Y\sin\alpha\sin\delta + \dot{Z}\cos\delta \end{aligned}$$

We then have:

$$\begin{aligned} \mu_\alpha \cos\delta &= A\cos 2\alpha + B\sin 2\alpha + C\sin\alpha + D\cos\alpha + E \\ \mu_\delta &= A'\cos 2\alpha + B'\sin 2\alpha + C'\sin\alpha + D'\cos\alpha + E' \end{aligned}$$

$$\begin{aligned} A &= (+0.204\,P\sin 2l_0 - 0.437\,P\cos 2l_0)\cos\delta \\ B &= (-0.157\,P\cos 2l_0 - 0.568\,P\sin 2l_0)\cos\delta \\ C &= (-0.150\,P\cos 2l_0 - 0.405\,P\sin 2l_0 + 0.150\,Q)\sin\delta \\ D &= (-0.862\,P\cos 2l_0 + 0.070\,P\sin 2l_0 + 0.862\,Q)\sin\delta \\ E &= +0.465\,Q\cos\delta \\ A' &= (-0.078\,P\cos 2l_0 - 0.284\,P\sin 2l_0)\sin 2\delta \\ B' &= (+0.218\,P\cos 2l_0 - 0.120\,P\sin 2l_0)\sin 2\delta \\ C' &= (-0.862\,P\cos 2l_0 + 0.070\,P\sin 2l_0)\cos 2\delta - 0.862\,Q \\ D' &= (+0.150\,P\cos 2l_0 + 0.405\,P\sin 2l_0)\cos 2\delta + 0.150\,Q \\ E' &= +0.580\,P\sin 2l_0\sin 2\delta \end{aligned}$$

The analysis of the proper motions has been made by giving equal weights to the hourly means, regardless of the number of stars in each hour. The solution gives in this case the different harmonics independent from each other.

The results from the stars of Gyllenberg's Catalogue are as follows:

TABLE 1

MAGNITUDE	<7.9	8.0–8.5	8.6–8.8	8.9–9.1	>9.1
A	−″.0069	−″.0068	−″.0062	−″.0043	−″.0029
B	+.0001	+.0004	+.0038	+.0017	+.0044
A'	+.0009	+.0023	−.0014	+.0029	−.0002
B'	+.0004	+.0030	+.0018	+.0020	+.0044
No. of stars	1885	2426	2347	2700	2013

From the tabulated values I derived: from the μ_α components for our five classes of magnitudes:

$P\sin 2l_0$	−″.0044	−″.0050	−″.0111	−″.0061	−″.0100
$P\cos 2l_0$	+.0172	+.0167	+.0122	+.0089	+.0033
l_0	353°	352°	339°	343°	324°
P	.018	.017	.016	.011	.010

from the μ_δ components:

$P\sin 2l_0$	−″.0034	−″.0109	+″.0024	−″.0117	−″.0045
$P\cos 2l_0$	+.0003	+.0091	+.0097	+.0040	+.0190
l_0	317°	335°	367°	324°	353°
P	.003	.015	.010	.012	.019

A least-squares solution from the proper motions in right ascension and in declination combined gave:

$P \sin 2l_0$	$-''.0041$	$-''.0066$	$-''.0076$	$-''.0075$	$-''.0089$
$P \cos 2l_0$	$+.0091$	$+.0111$	$+.0082$	$+.0056$	$+.0059$
l_0	348°	345°	339°	333°	332°

The mean values obtained from averaging $P \sin 2l_0$ and $P \cos 2l_0$ with equal weights for all magnitudes are $P \sin 2l_0 = -0''.0069$, $P \cos 2l_0 = +0''.0080$, from which $P = 0''.0106$, $l_0 = 340°$.

The direction of the center of rotation runs from 348° to 332°, decreasing regularly for the fainter stars. If we consider this as accidental the mean error of the mean value is $\pm 3°$, which, of course, is the internal mean error. The results for all the different magnitudes may be systematically in error by some defect in the catalogue places, although it is not easy to account for an error in the observations, both in declination and right ascension with a period of twelve hours. And, at any rate, the fact that the entirely independent data from μ_α and μ_δ components give consistent values for P and l_0 makes the rotation-hypothesis as a means of explaining the character of the proper motions highly probable.

On the other hand, it would not be surprising if the gradual decrease in l_0 were real. This would mean that the rotational center is slightly different for stars of different spectral types. The direction l_0 is the direction of the normal to the mean space velocities. The center of the globular cluster system has been determined by Shapley to be in the direction of 325°. Lindblad (loc. cit.) assumes that the direction of the center of the galactic system is at right angles to the direction of the asymmetrical drift, namely, at 330°. Oort (loc. cit.) has really found extremely strong evidence that the rotational center is in the same direction as the center of the globular cluster system as outlined by Shapley, although the distance from the rotational center is greatly at variance with Shapley's estimate of the distances of the globular clusters.

The direction of the normal to the stream motion here found to be toward 340° galactic longitude, deviates too much from 325° to be accounted for by accidental error. Thus, if the present results are not seriously vitiated by systematic errors, it remains an open question whether the center of the stellar system should be adopted at a larger galactic longitude or whether there is an appreciable difference between the direction of this center and the direction of the normal to the mean space velocities. The latter would indicate that the stars here under discussion show a radial component in their mean velocities, which thus would be not circular but spiral. However, I prefer to delay going into the details of this point until this discussion of the 50° to 55° zone in a subsequent paper.

Aside from the second harmonics, the rotation introduces a first harmonic and a constant term in μ_α and μ_δ. It is not possible to discuss these

terms definitely before we know how large the effect on the precession constant should be. It is evident that a term in the precession constant will introduce no second harmonics in the proper motions but the periodic part of μ_δ will be affected by it so as to introduce a presumably small change in the parallactic motion.

It is, however, worthwhile to investigate the influence of the assumed rotation on the parallactic motions of stars in a certain zone of declination even without considering the influence of an improved precession constant.

For the solution of the parallactic motion we have now:

$$\mu' = A\cos 2\alpha + B'\sin 2\alpha + (K\cos D_0\cos A_0 + C)\sin\alpha + (-K\cos D_0\sin A_0 + D)\cos\alpha + E$$

$$\mu'' = A'\cos 2\alpha + B'\sin 2\alpha + (K\sin\delta\cos D_0\sin A_0 + C')\sin\alpha + (K\sin\delta\cos D_0\cos A_0 + D')\cos\alpha - K\cos\delta\sin D_0 + E'$$

and the components

$$\begin{aligned} K\cos D_0\cos A_0 &= -X \\ K\cos D_0\sin A_0 &= -Y \\ K\sin D_0 &= -Z \end{aligned}$$

If we denote these components as computed without taking into account the rotational effect by X_α and Y_α if they have been computed from μ_α components, and by X_δ and Y_δ if they have been derived from μ_δ components, we have:

$$\Delta X_\alpha = X - X_\alpha = C \qquad \Delta X_\delta = X - X_\delta = \frac{D'}{\sin\delta}$$

$$\Delta Y_\alpha = Y - Y_\alpha = -D \qquad \Delta Y_\delta = Y - Y_\delta = \frac{C'}{\sin\delta}$$

which are the corrections to the parallactic motions in X and Y. Also

$$X_\delta - X_\alpha = C - \frac{D'}{\sin\delta}$$

$$Y_\delta - Y_\alpha = -\frac{C'}{\sin\delta} - D,$$

which give the differences found between the components as derived from μ_α and μ_δ components. Such a difference has really been found for the stars under discussion. Since the Y component is by far the most prominent, and since, moreover, C and D' are small, Q being negative and small as will be seen later, I confine myself to the Y components. The stars used for the computation of Y_α and Y_δ are the same as those used in the rotation effect. The results are in table 2.

The last four groups are in striking agreement; for the stars brighter than 8.0 magnitude the difference $Y_\delta - Y_\alpha$ is much larger. The mean value from the 9486 stars not brighter than 8.0 magnitude is $+0''.0078$; from all stars included it is $+0''.0092$. Using the value $P = +0''.0106$

above found, we find from $Y_\delta - Y_\alpha = -\frac{C'}{\sin\delta} - D = +0''.0078$; $Q = +0''.0002$. If $Y_\alpha - Y_\delta = +0''.0092$, $Q = +0''.0018$.

TABLE 2

MAGNITUDE	Y_α	Y_δ	$Y_\delta - Y_\alpha$	NO.
6.0–6.9	+".026	+".042	+".016	468
7.0–7.9	+.023	+.037	+.014	1302
8.0–8.5	+.014	+.020	+.006	2426
8.6–8.8	+.010	+.019	+.009	2347
8.9–9.1	+.011	+.018	+.007	2700
>9.1	+.006	+.015	+.009	2013

The quantity Q/P as determined from Oort's results is -0.72, and by means of his value of P, derived from the radial velocities and mean parallaxes of distant stars, $Q = -0''.0045$. For $Q = -0''.004$, $Y_\delta - Y_\alpha$ would be $+0''.004$ and the larger value here found if due to a rotation effect indicates a value of Q very nearly zero. Let us adopt the value $Q = -0''.004$ for the present. The corrections to the Y components of the parallactic motion then are:

$$\Delta Y_\delta = -".008\frac{\cos 2\delta}{\sin\delta} + \frac{.003}{\sin\delta}$$

$$\Delta Y_\alpha = +".011\sin\delta$$

In taking the mean, ΔY_δ has the weight $\sin^2\delta$, as compared with ΔY_α, and I find consequently the following corrections for different declinations in the northern sky. The signs are opposite for the southern sky.

δ	$\sin^2\delta\Delta Y_\delta$	ΔY_α	$\frac{\sin^2\delta\Delta Y_\delta + \Delta Y_\alpha}{1+\sin^2\delta}$
0	".000	".000	".000
15	−.001	+.003	+.002
30	−.000	+.005	+.004
45	+.002	+.008	+.007
60	+.006	+.010	+.009
75	+.010	+.011	+.011
90	+.011	+.011	+.011

For stars north of $+30°$ and south of $-30°$, the difference between the mean parallactic motions is of the order of $0''.02$. This is just the difference found by Perrine[6] between the far northern and far southern B-stars. The asymmetry in the proper motions of stars in the northern and southern hemispheres as emphasized by Perrine can thus be accounted for.

[1] *Astron. J.*, in press.

[2] *Publ. Astron. Soc. Pacific*, **39**, 203, 1927.

[3] Katalog von 11,800 Sternen der Zone +35° bis +40° AG Lund.

[4] *Arkiv för Matematik, Astronomi Voch Fysik*, **19A**, No. 21, 1925.

[5] *Bull. Astron. Institutes Netherlands*, **120**, 1927.

[6] *Monthly Notices R. A. S.*, **87**, 426, 1926.

REMARK ON A THEOREM OF R. L. MOORE

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In his paper "Concerning Indecomposable Continua and Continua Which Contain No Subsets That Separate the Plane,"[1] Prof. R. L. Moore showed the following theorem:

"If, in a Euclidean space of two dimensions, M is a bounded indecomposable continuum and G is a countable upper semi-continuous collection of mutually exclusive subcontinua of M and no continuum of this collection disconnects M then their sum does not disconnect M."

The purpose of this paper is to show that: (1) the following three restrictions are superfluous:

(*a*) The collection is *upper semi-continuous*

(*b*) The subcontinua are *mutually exclusive*

(*c*) *No continuum of the collection disconnects M.*

(2) The theorem does not hold true (in the space of three dimensions), if the condition that M is *bounded* be omitted; (3) however this condition may be omitted, if we replace the stipulation that the collection is *countable* by the stronger one that it is *finite.*

I. THEOREM (α). *If M is a bounded indecomposable continuum and G is a countable collection of subcontinua of M, then the sum of these sub-continua does not disconnect M.*

Proof.—Clearly we may omit the case in which M itself belongs to the collection G. By a theorem of Janiszewski and Kuratowski[2] an indecomposable continuum is the sum of an uncountable infinity of *"composants."*[3] Since each subcontinuum of M, different from M, is contained in one composant and since the collection G is countable, it follows that there exists a composant of M, call it P, such that $P \subset M-S$, where S denotes the sum of all the subcontinua of the collection G. As M is bounded, we have $\overline{P} = M$.[4] Therefore, $P \subset M-S \subset \overline{P}$ and, since P is connected, it follows, by a theorem of Hausdorff,[5] that $M-S$ is connected.

II. EXAMPLE. Consider the indecomposable continuum $\mathfrak{H}$ of our paper, "Sur les continus non-bornés."[6] This continuum is contained in the three-dimensional space. The set of its points $(x, 0, z)$ such that $0 < x < {}^{10}/_{9}$ is closed and is the sum of a countable infinity of mutually exclusive (unbounded) continua. It clearly disconnects the continuum $\mathfrak{H}$.

Thus, if the restriction that M is *bounded* is omitted, the theorem (α), in the case of space of three dimensions, is false. The case of the *plane* remains unsettled.

III. AUXILIARY THEOREM. *If $A_1, A_2, \ldots A_n$ are connected subsets of a connected set M such that*

$$M - (A_1 + A_2 + \ldots + A_n) = B + C \tag{1}$$

where

$$B \text{ and } C \text{ are mutually separated} \tag{2}$$

then the set $A_1 + A_2 + \ldots + A_n + B$ is composed of n (not necessarily different) connected sets.[7]

Proof.—Let

$$A_1 + A_2 + \ldots + A_n + B = D_1 + D_2 + \ldots + D_n \tag{3}$$

where

$$D_1, D_2, \ldots D_n \text{ are mutually separated (non-vacuous) sets.} \tag{4}$$

We shall show that D_1 is connected.

By (1) and (3):

$$M = A_1 + A_2 + \ldots A_n + B + C = D_1 + (D_2 + \ldots + D_n + C) \tag{5}$$

and, by virtue of (4), the sets D_1 and $(D_2 + \ldots + D_n)$ are mutually separated. Since M is connected, it follows, by (5), that D_1 and C are not mutually separated, for otherwise M would be decomposed into two mutually separated sets D_1 and $(D_2 + \ldots + D_n + C)$. Now, by (2), B and C are mutually separated and, since D_1 and C are not separated, it follows that D_1 is not contained in B. Therefore, we have by virtue of (3): $D_1 \cdot (A_1 + A_2 + \ldots + A_n) \neq 0$. Hence there exists an integer i such that $D_1 \cdot A_i \neq 0$. It follows that D_1 contains A_i entirely, for A_i is connected and is contained, by (3) and (4), in the sum of the mutually separated sets $D_1, D_2, \ldots, D_n$.

Similarly, if k is any integer $1, 2, \ldots, n$, then D_k contains one of the sets $A_1, A_2, \ldots, A_n$; since the sets $D_1, D_2, \ldots, D_n$ are, by (4), mutually exclusive, A_k cannot be contained in more than one of these sets. Clearly we may assume that

$$A_i \subset D_i \text{ for every integer } i = 1, 2, \ldots, n. \tag{6}$$

It follows that $A_2 + \ldots + A_n \subset D_2 + \ldots + D_n$ and since, by (4), $D_1 \cdot (D_2 + \ldots + D_n) = 0$, we have

$$D_1 \cdot (A_2 + \ldots + A_n) = 0. \tag{7}$$

Now, suppose D_1 is not connected. Hence

$$D_1 = E + F. \tag{8}$$

$$E \text{ and } F \text{ are mutually separated (non-vacuous) sets} \tag{9}$$

By (6) and (8) we have $A_1 = A_1 \cdot E + A_1 \cdot F$. Since A_1 is connected, it

follows from (9) that either $A_1 \cdot E = 0$ or $A_1 \cdot F = 0$. Clearly we may assume that $A_1 \cdot E = 0$. On the other hand $(A_2 + \ldots + A_n) \cdot E = 0$ as follows from (7) and (8). Therefore

$$(A_1 + A_2 + \ldots + A_n) \cdot E = 0. \qquad (10)$$

Since by (8) and (3): $E \subset D_1 \subset A_1 + A_2 + \ldots + A_n + B$, it follows from (10) that $E \subset B$. Hence, by (2), E and C are mutually separated. Now, E and F are mutually separated by (9) and E and $(D_2 + \ldots + D_n)$ are mutually separated by (8) and (4).

So that the sets E and $(F + D_2 + \ldots + D_n + C)$ are separated, contrary to hypothesis that M is connected, for we have by (8) and (5): $M = E + (F + D_2 + \ldots + D_n + C)$.

Thus the supposition that D_1 is not connected leads to a contradiction. Hence D_1 is connected. Similarly, if i is any integer $2, \ldots, n$, D_i is connected.

The truth of the auxiliary theorem is, therefore, established.

COROLLARY. *If $A_1, A_2, \ldots, A_n$ are subcontinua of a continuum M such that $M - (A_1 + A_2 + \ldots + A_n) = B + C$, two mutually separated sets, then $A_1 + A_2 + \ldots + A_n + B$ is composed of n (not necessarily different) continua.*

THEOREM (β). *If M is an indecomposable continuum (bounded or unbounded) and G is a finite collection of subcontinua of M, then the sum of these subcontinua does not disconnect M.*

Proof.—Suppose the sum S of these subcontinua disconnects M. Let $M - S = B + C$, two mutually separated non-vacuous sets. Then, by the corollary, $S + B$ and $S + C$ are composed of a finite number of continua and since $B \neq O \neq C$ it follows that $S + B \neq M \neq S + C$, so that each of these continua is different from M. Thus M is the sum of a finite number ($\geqslant 2$) of continua (different from M). But this is a contradiction of the hypothesis that M is an indecomposable continuum.[8]

[1] *Proc. Nat. Acad. Sci.*, **12**, 1926, p. 362. A continuum is said to be *indecomposable* if it is not the sum of two continua neither of which is identical with it.

[2] "Sur les continus indécomposables," *Fund. Math.*, **1**, 1920, p. 221. Cf. S. Mazurkiewicz, *Fund. Math.*, **10**, 1927.

[3] A "composant" of a point p of an indecomposable continuum M is the sum of all continua X such that: $p \subset X \subset M$, $X \neq M$. See *Fund. Math.*, **1**, p. 218; cf. the Brouwer's term "nerve," *Proc. K. Akad. Wett., Amsterdam*, **1911.**

[4] *Fund. Math.*, **1**, p. 223, Theor. VII. $\overline{X}$ denotes the set X plus all its limit points.

[5] *Grundzüge der Mengenlehre*, Leipzig, **1914**, p. 246, Theor. IV.

[6] *Fund. Math.*, **5**, 1924, p. 53. Each composant of $\mathfrak{H}$ is a continuum.

[7] For the case $n = 1$, this theorem was proved in our paper, "Sur les ensembles connexes," *Fund. Math.*, **2**, 1921, p. 210, Theor. VI; for $n = 2$, see G. T. Whyburn. *Fund. Math.*, **10**.

[8] *Fund. Math.*, **1**, p. 216, Theor. III.

CONCERNING THE OPEN SUBSETS OF A PLANE CONTINUOUS CURVE[1]

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A subset N of a point set M is an open subset of M provided the point set $M-N$ is either vacuous or closed. If N is an open subset of the set M, the boundary of N with respect to M[2] is the set of all those points of $M-N$ which are limit points of N. In this paper the term *M-boundary* will be used to denote the boundary of a point set with respect to a given point set M. A point set M is said to be *cyclicly connected*[3] provided that every two points of M lie together on some simple closed curve which is contained in M. A boundary point P of a point set R will be said to be *regularly accessible from* R provided it is true that for every positive number ϵ there exists a positive number δ_ϵ such that every point X of R whose distance from P is less than δ_ϵ can be joined to P by an arc XP which lies, except for the point P, wholly in R and which is of diameter less than ϵ. A point set M is said to have property S[4] provided it is true that for every positive number ϵ, M is the sum of a finite number of connected point sets each of diameter less than ϵ.

1. ACCESSIBILITY THEOREMS. THEOREM 1. *In order that every point of the M-boundary, B, of a bounded connected open subset, R, of a continuous curve, M, should be regularly accessible from R it is necessary and sufficient that R should have property S.*

Proof.—The condition is sufficient. For suppose R has property S. Let P be any point of B and ϵ any positive number. Let C be a circle with center P and radius $\epsilon/2$. Let R be expressed as the sum of a finite number of connected point sets $K_{11}, K_{12}, K_{13}, \ldots, K_{1n_1}$, each of diameter $<\epsilon/4$. Then there exists a number $\delta_\epsilon > 0$ such that every point of R whose distance from P is less than δ_ϵ lies in some one of the sets $K_{11}, K_{12}, \ldots, K_{1n_1}$ which has P for a limit point. I shall show that every such point of R can be joined to P by an arc of diameter $< \epsilon$ and which lies, except for the point, P, wholly in R. Let X denote one such point of R. Let R_0 denote the maximal connected subset of R which contains X and lies within C. Then R_0 is an open subset of M which has P in its M-boundary. Now let R be expressed as the sum of a finite number of connected point sets $K_{21}, K_{22}, K_{23}, \ldots, K_{2n_2}$, each of diameter $< \frac{1}{2}\delta_\epsilon$ and also less than $\frac{1}{2}$ the diameter of R_0. At least one of the sets $K_{21}, K_{22}, \ldots, K_{2n_2}$, say K_{2j}, lies in R_0 and has P for a limit point. Let R_1 denote the maximal connected subset of R which contains K_{2j} and has the property that every one of its points is at a distance $<\frac{1}{2}\delta_\epsilon$ from P. Then R_1 is an open subset

of M which lies in R_0 and has P in its M-boundary. Now let R be expressed as the sum of a finite number of connected point sets $K_{31}, K_{32}, \ldots, K_{3n_3}$, each of diameter $< \frac{1}{4}\delta_\epsilon$ and also less than $\frac{1}{2}$ the diameter of R_1. There exists a set K_{3j} which lies in R_1 and has P for a limit point. Let R_2 denote the maximal connected subset of R which contains K_{3j} and has the property that every one of its points is at a distance from P less than $\frac{1}{4}\delta_\epsilon$. Now let R be expressed as the sum of a finite number of connected point sets $K_{41}, K_{42}, \ldots, K_{4n_4}$, each of diameter $< \frac{1}{8}\delta_\epsilon$ and also less than $\frac{1}{2}$ the diameter of R_2, and so on. This process may be continued indefinitely, and thus we obtain an infinite sequence of connected open subsets of R: $R_0, R_1, R_2, R_3, \ldots$, such that for every integer $n \geq 0$, R_{n+1} has P in its M-boundary and is a subset of R_n, and such that the diameter of R_n approaches zero as n increases indefinitely.

For each integer $n > 0$, let X_n denote a point belonging to R_n. Since for every $n \geq 0$, R_n is a connected open subset of M, then by a theorem of R. L. Moore's,[5] R_n is arcwise connected. Hence, there exists in R_0 an arc XX_1, and for every integer $n > 0$, R_n contains an arc X_nX_{n+1}. It is easy to see that the point set $P + XX_1 + X_1X_2 + X_2X_3 + \ldots$ is closed and that it contains an arc XP such that $XP - P$ lies in R_0. Therefore, since R_0 is of diameter $\leqq \epsilon$, and since X is any point of R whose distance from P is less than δ_ϵ, it follows that every point P of B is regularly accessible from R.

The condition is also necessary. For suppose that every point of B is regularly accessible from R, but suppose, contrary to this theorem, that R does not have property S. Then for some positive number ϵ, R cannot be expressed as the sum of any finite collection of connected point sets each of which is of diameter less than ϵ. For each point P of R, let G_p denote the set of all those points of R which can be joined to P by a connected subset of R of diameter less than $\epsilon/2$. Let G denote the collection of connected point sets G_p thus obtained. Then since R is not a subset of the sum of the elements of any finite subcollection of G, it easily follows that R contains an infinite sequence of points $P_1, P_2, P_3, \ldots$, having a sequential limit point P and such that no two elements of this sequence can be joined by a connected subset of R of diameter $< \epsilon/2$. Let C_1 denote a circle with P as center and of diameter $\epsilon/4$. Let C_2 be a circle concentric with C_1 and of diameter $\epsilon/8$. There exists an integer k such that for every integer $i > k$, P_i lies within C_2. Let Q be a point of R which is without C_1. Then for every integer $i > k$, R contains an arc[5] P_iQ. Each arc P_iQ contains an arc P_iQ_i, where Q_i is some point on C_1, such that $P_iQ_i - Q_i$ lies within C_1. Each arc P_iQ_i contains an arc t_i containing a point A_i in common with C_1 and a point X_i in common with C_2 and such that $t_i - (A_i + X_i)$ lies in the domain between C_1 and C_2. Clearly no two of these arcs $[t_i]$ can have a common point. Since R is bounded, the

sequence of arcs $[t_i]$ contains an infinite subsequence t_{n_1}, t_{n_2}, t_{n_3}, . . . , where $n_1 > 1$, which has a sequential limiting set T which is a continuum belonging to B and having at least one point A in common with C_1 and at least one point X in common with C_2 and such that A and X are sequential limit points of the sequences $[A_{n_i}]$ and $[X_{n_i}]$, respectively. Let r denote the point set $r_1 + r_2 + t_1$, where r_1 is a radius of C_2 drawn to the point X_1, and r_2 is a radial ray of C_1 emerging from the point A_1. Since by hypothesis every point of B is regularly accessible from R, it follows that there exists an integer j such that for every integer $i > j$, A_{n_i} can be joined to A and X_{n_i} to X by arcs a_i and x_i, respectively, each of diameter $> \epsilon/30$ and also less than the minimum distance between the sets r and t_{n_i}, and such that, for every such i, $a_i - A$ and $x_i - X$ belong to R. For each integer $i > j$, the point set $t_i + a_i + x_i$ contains an arc T_i from A to X. Each pair of the arcs T_i have in common exactly the two points A and X. Hence the sum of each such pair of arcs is a simple closed curve. Since no one of these curves has a point in common with r, it follows that the point P is without every one of them. Furthermore, P is not a limit point of the sum of these curves. Therefore, since P is the sequential limit point of the sequence of points P_{n_1}, P_{n_2}, P_{n_3}, . . . , there exists an integer m such that the points P_{n_m}, $P_{n_{m+1}}$, and $P_{n_{m+2}}$ are without every one of these curves. Of the three arcs T_{n_m}, $T_{n_{m+1}}$, and $T_{n_{m+2}}$, one of them, say T_{n_m}, must lie, except for the points A and X, wholly within the simple closed curve J formed by the sum of the other two. Hence the arc $P_{n_m}Q_{n_m}$ contains a point within J and also a point P_{n_m} without J, and, therefore, it contains a point Y in common with J. The point Y cannot be either A or X, for $P_{n_m}Q_{n_m}$ belongs to R, and the points A and X belong to B. Hence Y belongs either to $T_{n_{m+1}} - (A + X)$ or to $T_{n_{m+2}} - (A + X)$, say to $T_{n_{m+1}} - (A + X)$. But the set of points $P_{n_m}Q_{n_m} + T_{n_m} - (A + X) + T_{n_{m+1}} - (A + X) + P_{n_{m+1}}Q_{n_{m+1}}$ is then connected, a subset of R, and of diameter less than ϵ, and it contains two points of the sequence P_1, P_2, P_3, . . . But this is contrary to the definition of the sequence P_1, P_2, P_3, . . . Thus the supposition that R does not have property S leads to a contradiction.

THEOREM 2. *In order that every point of the boundary, B, of a bounded two-dimensional domain, D, should be regularly accessible from D it is necessary and sufficient that every point of B should be accessible from D from all sides in the sense of Schoenflies.*[6]

Since two-dimensional space is a plane continuous curve, and since theorem 1 holds for any plane continuous curve, bounded or not, theorem 2 is an immediate consequence of theorem 1 and of the following theorem of the author's:[7] *In order that a bounded two-dimensional domain, D, should*

have property S it is necessary and sufficient that every point of the boundary of D should be accessible from D from all sides in the sense of Schoenflies.

THEOREM 3. *If R is a connected open subset of a bounded continuous curve, M, every subcontinuum of which is a continuous curve, then every point of the M boundary of R is regularly accessible from R.*

Proof.—By a theorem of the author's,[8] R has property S. Hence, by theorem 1, every point of the M-boundary of R is regularly accessible from R.

2. THEOREM 4. *If M and N are bounded continuous curves, N being a subset of M, R is any maximal connected subset of $M - N$, and B is the M-boundary of R, then (a) every maximal connected subset of B is a continuous curve, and (b) for every positive number ϵ, not more than a finite number of maximal connected subsets of B are of diameter greater than ϵ.*

Proof.—Let K denote the boundary of that complementary domain of N which contains R. Then B is a subset of K. Therefore, since by two theorems of R. L. Wilder's,[9] every subcontinuum of K is a continuous curve and, for any $\epsilon > 0$, K cannot contain more than a finite number of mutually exclusive continua each of diameter $> \epsilon$, it follows that every continuum which B contains is a continuous curve and that not more than a finite number of maximal connected subsets of B are of diameter greater than any preassigned positive number.

In connection with the proof just given, it is to be noted that if R is any connected open subset of any continuum M, then the M-boundary B of R is also the boundary of an ordinary two-dimensional domain, namely, of that complementary domain of B which contains R. Therefore, we immediately obtain many important properties of such point sets B. For example, if as in the case of theorem 4, above, B is a subset of a continuous curve which contains no point of R, we know immediately that (1) every closed and connected subset of B is a continuous curve, (2) every connected subset of B is arcwise connected, and (3) for any $\epsilon > 0$, B contains at most a finite number of mutually exclusive continua,[10] each of diameter $> \epsilon$. In particular, it follows that if M is a continuous curve every subcontinuum of which is a continuous curve, the M-boundary of *every* connected open subset of M has property (2) just mentioned. Furthermore, if M is any continuous curve and B has properties (*a*) and (*b*) of theorem 4, then B also has properties (1), (2) and (3) above mentioned.

THEOREM 5. *If M and N are bounded continuous curves, N being a subset of M, and R is any maximal connected subset of $M - N$, then R has property S.*

Proof.—Suppose, on the contrary, that R does not have property S. Then for some positive number ϵ, it is impossible to express R as the sum of any finite collection of connected point sets each of diameter $< \epsilon$. Let B denote the M boundary of R. Then B is a subset of N. Let the points

P, P_1, P_2, P_3, . . . , the circles, C_1 and C_2, the integer, k, the arcs $[P_iQ]$, $[P_iQ_i]$, $[t_i]$, and $[t_{n_i}]$, the continuum T, the set r, and the points A, X, $[A_i]$, and $[X_i]$, be selected and defined just as was done in the second part of the proof of theorem 1. Now since M and N are regular and each contains the points A and X, there exist circles C_a and C_x with centers A and X, respectively, and such that every point of M or of N lying within C_a or C_x can be joined to A or to X, respectively, by an arc of M or by an arc of N which is of diameter less than $\epsilon/30$ and which has no point in common with r. Furthermore, there exist circles $C_{a'}$ and $C_{x'}$, concentric with C_a and C_x, respectively, and such that every point of M which is within $C_{a'}$ or $C_{x'}$ can be joined in M to A or to X, respectively, by an arc which lies wholly within C_a or C_x, respectively. Now since A and X are sequential limit points of the sequences $[A_{n_i}]$ and $[X_{n_i}]$, respectively, there exists an integer j such that for every integer $i > j$, A_{n_i} lies within $C_{a'}$ and X_{n_i} lies within $C_{x'}$. Then for every integer $i > j$, M contains arcs $A_{n_i}A$ and $X_{n_i}X$ each of diameter $< \epsilon/30$ which lie within C_a and C_x, respectively, and neither of which has a point in common with r. For each $i > j$, on the arcs $A_{n_i}A$ and $X_{n_i}X$, respectively, in the order from A_{n_i} to A and from X_{n_i} to X, respectively, let c_{n_i} and d_{n_i}, respectively, denote the first points belonging to N. Then, for each $i > j$, the sum of the arcs t_{n_i}, $A_{n_i}c_{n_i}$ (of $A_{n_i}A$) and $X_{n_i}d_{n_i}$ (of $X_{n_i}X$) contains an arc T_i from c_{n_i} to d_{n_i} which lies, except for the points c_{n_i} and d_{n_i}, wholly in R. There exists in N arcs $c_{n_i}A$ and $d_{n_i}X$ each of diameter $< \epsilon/30$ and neither of which has a point in common with r. Since T is a closed and connected subset of B, it follows by theorem 4 and the discussion immediately following theorem 4 that T is a continuous curve. Hence T contains an arc t from A to X. For each $i > j$, the sum of the arcs $c_{n_i}A$, $d_{n_i}X$ and t, contains an arc E_i from c_{n_i} to d_{n_i}. For each $i > j$, E_i is a subset of N, and $E_i + T_i$ is a simple closed curve J_i.

Now since no one of the curves J_i has a point in common with r, it follows that P is without every one of these curves. Furthermore, P is not a limit point of their sum. Therefore, there exists an integer m greater than j and k such that for every integer $n \geq m$, P_n is without every one of the curves $[J_i]$. Now the set of points $T_{m+1} - (c_{n_{m+1}} + d_{n_{m+1}})$ lies wholly either within or without the curve J_m. If it lies within J_m, then the arc $P_{n_{m+1}}Q_{n_{m+1}}$, having a point within J_m and a point $P_{n_{m+1}}$ without J_m, must contain a point Y of J_m. The point Y must belong to $T_m - (c_{n_m} + d_{n_m})$. Hence, in this case, the set of points $P_{n_{m+1}}Q_{n_{m+1}} + P_{n_m}Q_{n_m} + T_m - (c_{n_m} + d_{n_m})$ is a connected subset of R of diameter less than ϵ and containing two points of the sequence P_1, P_2, P_3, . . . , con-

trary to the definition of this sequence. And if, on the other hand, the set $T_{m+1} - (c_{n_{m+1}} + d_{n_{m+1}})$ lies without J, then either the set $T_m - (c_{n_m} + d_{n_m})$ lies within the curve J_{m+1} or else the sum of the arcs T_m, T_{m+1}, $c_{n_m}A$, $d_{n_m}X$, $c_{n_{m+1}}A$, and $d_{n_{m+1}}X$ contains a simple closed curve J which enclosed a point Z of T. In the former case, we are led to a contradiction in the same manner as in the case just treated. In the latter case, since Z is on the limiting set of the sequence of arcs t_{n_1}, t_{n_2}, . . . , there exists an integer $b > m + 1$ such that the arc P_bQ_b contains a point within J. And since P_b lies without J, P_bQ_b must contain a point Y in common with J. The point Y must belong either to $T_m - (c_{n_m} + d_{n_m})$ or to $T_{m+1} - (c_{n_{m+1}} + d_{n_{m+1}})$, and either of these cases leads to a contradiction in a manner similar to that described in the case treated above. Thus the supposition that R does not have property S leads to a contradiction, and the theorem is proved.

I will remark that under the hypothesis of theorem 5, it follows immediately by theorems 5 and 1 that every point of the M-boundary of R is regularly accessible from R.

THEOREM 6. *If the connected open subset, R, of a continuous curve, M, has property S, and B denotes the M-boundary of R, then (a) every maximal connected subset of B is a continuous curve, and (b) if ϵ is any positive number, not more than a finite number of maximal connected subsets of B are of diameter greater than ϵ.*

Theorem 6 may be proved by a method not essentially different from that used by R. L. Moore to prove theorem 4 of his paper "Concerning Connectedness im kleinen and a Related Property."[4]

3. AN EXAMPLE. *There exists a connected open subset, R, of a continuous curve, M, such that (1) every maximal connected subset of B, the M-boundary of R, is a continuous curve, (2) only one maximal connected subset of B is of diameter > 0, (3) every point of B is accessible from R, (4) R does not have property S, and (5) $R + B$ is not a continuous curve.*

Let S denote the rectangle with vertices $(1, -1)$, $(1, 2)$, $(-1, 2)$ and $(-1, -1)$; let S' denote the square with vertices $(0, 0)$, $(1, 0)$, $(1, 1)$ and $(0, 1)$; and let G denote the square with vertices $(0, 0)$, $(0, 1)$ $(-1, 1)$, and $(-1, 0)$, plus its interior. Let L_n, $(n = 1, 2, 3, \ldots)$, be defined by the relations $x = 1/2^n$, $0 \leq y \leq 1$. Let H_n, $(n = 1, 2, 3, \ldots)$, be the sum of the intervals cut out between the line $x = 0$ and L_n on n lines which are parallel to the line $y = 0$ and which are equally spaced between the lines $y = 0$ and $y = 1$. Let

$$M = S + S' + G + \sum_{n=1}^{\infty} L_n + \sum_{n=1}^{\infty} H_n.$$

For each positive integer n, let I_n be the sum of n intervals so chosen that

one of these intervals lies on each interval of H_n and so that each of them lies between L_n and L_{n+1} and has no point in common with either L_n or L_{n+1}. Let $K = I_1 + I_2 + I_3 + \ldots$, plus the straight-line interval from (0, 0) to (0 ,1). Let $R = M - K$, and let B denote the M-boundary of R. Then M is a continuous curve, and R is a connected open subset of M. And since B consists of the straight-line interval from (0, 0) to (0, 1) plus a countable set of points whose accumulation set is this interval, it is readily seen that B satisfies conditions (1) and (2) above, and that every point of B is accessible from R. But clearly R does not have property S, and $R + B$ is not a continuous curve.

THEOREM 7. *If the M-boundary B of a connected open subset R of a bounded continuous curve M either (1) is regular or (2) has no continuum of condensation, then (I) R has property S and (II) R + B is a continuous curve.*

Theorem 7 may be proved by methods very similar to those used in the proofs of theorems 1 and 5, together with the use of the obvious fact that if any point set has property S, then that set plus its limit points must also[11] have property S.

4. CYCLICLY CONNECTED SUBSETS. THEOREM 8. *In order that a connected open subset of a continuous curve should be cyclicly connected it is necessary and sufficient that it should have no cut point.*

Theorem 8 may be proved by the same method as was used to prove theorem 1 of my paper "Cyclicly Connected Continuous Curves."[3]

THEOREM 9. *If the point set H is cyclicly connected and H′ (H plus its limit points) is regular, then H′ is cyclicly connected.*

Theorem 9 is a direct consequence of theorem 1 of my paper "Cyclicly Connected Continuous Curves."[3]

THEOREM 10. *If the M-boundary B of a connected open subset, R, of a cyclicly connected continuous curve M is connected and R + B is regular, then R + B is cyclicly connected.*

Proof.—No point of B is a cut point of the curve $R + B$. And since M is cyclicly connected and, hence,[3] has no cut point, it follows that if X is any point of R, then B must contain at least one limit point of each maximal connected subset of $R - X$. Therefore, since B is connected, for every point X of R, the set $(R + B) - X$ is connected. Hence the curve $R + B$ has no cut point and, therefore,[3] it is cyclicly connected.

THEOREM 11. *Let C and M be cyclicly connected continuous curves, and let C be a subset of M. Let $[R_n]$, $n = 1, 2, 3, \ldots$, be the collection of all the maximal connected subsets of $M - C$. Then (1) for every positive integer i, $R_i + C$ is a cyclicly connected continuous curve, and (2) if $[R_{n_i}]$ is any subcollection of $[R_n]$, then $M - \sum_{i=1,2,3,\ldots} R_{n_i}$[12] is a cyclicly connected continuous curve.*

[1] Presented to the American Mathematical Society, Dec. 31, 1926, and Feb. 26, 1927.

[2] Moore, R. L., *Trans. Amer. Math. Soc.*, **21**, 1920 (345).

[3] Whyburn, G. T., "Cyclicly Connected Continuous Curves," these PROCEEDINGS, **13**, 1927 (31–38).

[4] Moore, R. L., *Fund. Math.*, **3**, 1921 (233–237).

[5] Moore, R. L., *Math. Zeit.*, **15**, 1922 (254–260).

[6] Schoenflies, A., *Die Entwickelung der Lehre von den Punktmannigfaltigkeiten*, Leipzig, **1908**, S. 237.

[7] Whyburn, G. T., "Concerning Continua in the Plane," *Trans. Amer. Math. Soc.*, **29**, 1927 (369–400).

[8] Whyburn, G. T., these PROCEEDINGS, **12**, 1926 (761–767).

[9] Wilder, R. L., "Concerning Continuous Curves," *Fund. Math.*, **7**, 1925 (340–377), theorems 11 and 6.

[10] For proofs of these and other related theorems for the case of the boundary of a complementary domain of a plane continuous curve, see Wilder, R. L., loc. cit.

[11] Cf., for example, the proof of theorem 4 of my paper cited in ref. 8. Use will also be made here of Sierpinski's theorem that any continuum having property, S, is a continuous curve. See Sierpinski, W., *Fund. Math.*, **1**, 1920 (44–66).

[12] That this set of points is a continuous curve has recently been proved by H. M. Gehman in his paper "Some Relations between a Continuous Curve and Its Subsets," *Annals of Math.* Ser. 2, **28**, 1927 (103–111), theorem 8.

ON THE FUNCTIONAL INDEPENDENCE OF RATIOS OF THETA FUNCTIONS

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1. In the theory of multiply periodic functions there is a theorem which for many reasons it would be desirable to prove directly. Existing proofs either are not completely general or are very roundabout.[1] It may be stated thus: *Consider a family of linearly independent Θ's of the same order and characteristic, in maximum number, attached to a definite period matrix, Ω. When the number of functions $>p$, their genus, then among their ratios to one of them there are p functionally independent.*

Let $\Theta_1, \Theta_2, \ldots, \Theta_r$ be the functions and $u_1, \ldots, u_p$, the variables. The above theorem is equivalent to proving that the matrix

$$\left\| \frac{\Theta_h}{\frac{\partial \Theta_h}{\partial u_s}} \right\| \tag{1}$$

is of rank $p + 1$ when the (u)'s are arbitrary. This we shall do under the assumption that the order n is sufficiently great. Apparently restrictive, it is really immaterial in most cases. The proof, merely outlined here, is, as we shall see, not only direct but also very elementary.

2. We take for Ω the well-known normal form:

$$||0, \ldots, 0,1,0, \ldots 0; a_{j1}, \ldots a_{jp}||; a_{jk} = a_{kj},$$

and for Θ's those of characteristic zero:

$$\Theta_h = \sum \exp [n\pi i \sum_{j,k} a_{jk}(m_j + c_j^h)(m_k + c_k^h) + 2 \sum_j (m_j + c_j^h)u_j];$$

$$(nc_j^h = q_j^h \text{ positive integer less than } n).$$

The first summation is with respect to the m's which run through all integral values from $-\infty$ to $+\infty$, while the q's are arbitrary within their limits. It will be observed that Ω is not the most general normal matrix. In the general case the proof is the same but the formulas a little more involved.

To establish our theorem it suffices to show that for fixed q's when the u's are all zero and $n \longrightarrow \infty$, $(1) \longrightarrow$ a matrix of rank $p+1$. We need, therefore, the limits of $\Theta_h(0)$, $\dfrac{\partial \Theta_h(0)}{\partial u_s}$, as $n \longrightarrow \infty$.

I say that the first expression $\longrightarrow 1$. Let a'_{jk} denote the imaginary part of a_{jk}. Since $\sum a'_{jk}x_jx_k$ is a definite, positive, quadratic form, there exists a positive w such that

$$\pi \sum a'_{jk}x_jx_k > w \sum x_j^2.$$

$$\therefore | \Theta_h(0) - 1 | < -1 + \sum \exp[-nw \sum (m_j + c_j^h)^2] =$$

$$-1 + \prod_j \sum_{m=-\infty}^{+\infty} \exp [-nw(m + c_j^h)^2].$$

The last exponential $\longrightarrow 1$ when $m = 0$, and $\longrightarrow 0$ for any other fixed value of m. Also for n greater than twice any q, and writing c for c_j^h, the ratio of that exponential to $e^{-nw|m|}$ is less than

$$\exp - nw[m^2 - (1 + 2c) \mid m \mid + c^2] < 1$$

when $| m | > 2$, since owing to $c < 1/2$, the bracket is always negative.

$$\therefore | \Theta_h(0) - 1 | < (1 + \epsilon + 2 \sum_{m=1}^{+\infty} e^{-\pi w m})^p - 1,$$

where $\epsilon \longrightarrow 0$ with $1/np$. As the sum just written is equal to

$$\frac{e^{-nw}}{1 - e^{-nw}} \longrightarrow 0 \text{ with } 1/n,$$

$\Theta_h(0) \longrightarrow 1$ as asserted.

Concerning the second expression whose limit is desired, we have

$$\left| \frac{\partial \Theta_h(0)}{\partial u_s} - i\pi q_s^h \Theta_h(0) \right| < n \sum m_s \exp - nw \sum_j (m_j + c_j^h)^2$$

which I say $\longrightarrow 0$. Much as above it comes down to showing that

$$n \sum_{m=1}^{+\infty} m e^{-nwm} \longrightarrow 0.$$

As $m < e^m$, this sum may be replaced by

$$n \sum e^{-(nw-1)m} < \frac{n}{e^{nw} - e} \longrightarrow 0.$$

Hence the limit of our expression is $i\pi q_s^h$.

3. Referring then to the matrix (1), among its determinants of order $p + 1$ there is one whose limit is

$$(i\pi)^p \begin{vmatrix} l \\ q_s^h \end{vmatrix}, \quad (h = 1, 2, \ldots p + 1; s = 1, 2, \ldots p).$$

Since after all the q's are arbitrary integers, we may so choose them that this determinant $\neq 0$. This proves our theorem.

[1] See the Encyklopädie article II B 7, Abelsche Functionen, by Krazer-Wirtinger, also the forthcoming bulletin of the National Research Council; selected topics in algebraic geometry, Ch. 17, §1.

THE DIRECTION OF EJECTION OF X-RAY ELECTRONS

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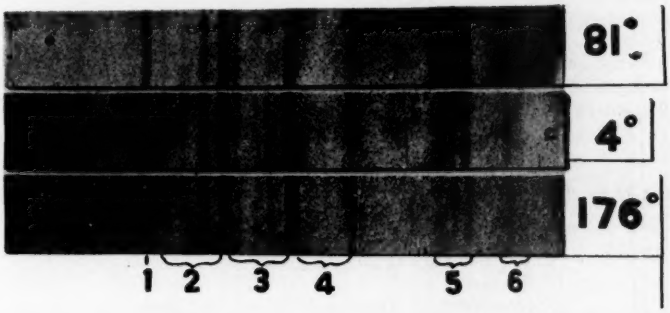
No conclusive evidence that the direction in which photo-electrons are ejected by X-rays depends in any way upon the nature of the atom from which the ejection takes place has as yet been brought forward. Auger,[1] using the C. T. R. Wilson cloud expansion-chamber method, showed that the most probable direction of ejection in a gas is a function of the frequency of the incident X-rays, but the variations which he found in this most probable direction with the nature of the gas used (oxygen or nitrogen, argon, krypton, xenon) were probably less than the experimental error, particularly as heterogeneous X-rays were used and the frequency of the X-rays which were most effective in ejecting electrons may have varied from gas to gas. Loughridge[2] concluded that the most probable direction of ejection was the same for water-vapor, air and argon, but the absorption

energies of even the K-shells of all these atoms is so small that at best only a small effect would be expected in these cases. Bothe[3] using the point-discharge ion-counter made observations on air and on gases the molecules of which contained I, Br and Cl atoms. His results show small variations with the nature of the gas, but again as heterogeneous X-rays were used and his maxima were not sharp the variations were probably less than the experimental accuracy.

If the electron orbits inside the atom, postulated by Bohr with such amazing success, have physical reality, then it is to be expected that the momentum of the electron in these orbits will have some effect upon the direction it will be moving when removed from the atom. In fact, all the attempts which have been made to take the momentum of the electron in its atomic orbit into account in a theory of the direction of ejection of photo-electrons have led to the conclusion that (for circular orbits at least) the most probable direction of ejection of an electron whose absorption energy (i.e., the energy necessary to remove it to infinity) is only a little less than the energy of the incident quantum will be greatly different (as much as 90°) from the most probable direction of ejection of an electron whose absorption energy is negligible compared to the energy of the incident quantum (see theories of Auger and Perrin[4] and of Bothe[5]). That this is not the case for electrons ejected from K-orbits has been proved by Auger[1] who found that with 45 k.v. X-rays the most probable direction of ejection of the K-electrons of xenon (whose absorption energy is 35 k.v.) was not appreciably different from that of the K-electrons of oxygen or nitrogen (whose absorption energy is negligible). In their recent theory of the space-distribution of X-ray electrons Auger and Perrin[4] point out that this constitutes a very serious difficulty for the old conception of electronic orbits and attempt to evade it by making use of the new wave-mechanics which permits the moment of momentum of the K-electron about the nucleus to be put equal to zero. The difficulty cannot be evaded in this way, however, as the writers have recently obtained evidence which proves that the most probable direction of ejection of the L_r as well as the K-electrons does not change as the absorption energy increases. The purpose of this paper is to present this evidence.

The method which was used so successfully by de Broglie, Whiddington and Robinson, to obtain magnetic spectra of the electrons ejected by X-rays, has recently been modified by one of us[6] in such a way as to make it possible to study the velocity and number of the electrons as a function of the angle of ejection. Magnetic spectra taken in this manner of electrons ejected from exceedingly thin metallic films show that the most probable direction of ejection is a little forward of perpendicular to the direction of the X-ray beam and that this most probable direction is approximately the same for all electrons whether from the K-, L_r-, M- or N-levels and

whether the absorption energy of the level is large or small. This is well shown in figure 1, in which magnetic spectra of the electrons emerging at three different angles from a thin film of gold traversed by primary X-rays from molybdenum are reproduced. The lines are all relatively strong in the spectrum taken at 81°, i.e., a little forward of perpendicular to the X-ray beam. At 4° (measured from the forward direction of the X-ray beam) all the lines, except those due to the fluorescent radiation of the gold itself which should be the same in all directions, are much weaker and at 176° they have practically disappeared. It is to be noted espe-



Magnetic spectra of electrons emerging at various angles from a thin sputtered film of gold traversed by primary x-rays from molybdenum

1. L_{III} electrons ejected by MoK_α
2. M electrons ejected by AuL_α,
3. M electrons ejected by AuL_β,
4. M electrons ejected by AuL_γ,
5. M electrons ejected by MoK_α
6. N&O electrons ejected by MoK_α

FIGURE 1

cially that the intensity of the line numbered 1, which is due to electrons from the circular L-orbit whose absorption frequency is 289×10^{16} as compared with a frequency of 421×10^{16} for the incident X-rays, falls off as the angle departs from 80° in practically the same way as do the intensities of the lines numbered 5 and 6 which are due to electrons from the outer orbits whose absorption frequencies are small.

The theory of Auger and Perrin demands that the most probable direction of ejection of the L_{III} electrons of gold by the K_α lines of molybdenum be in the neighborhood of 35°. A number of spectra of the L_{III} line have been taken at small intervals between 0° and 90° and it is certain that the

intensity is greatest close to 80° and that it falls off regularly as the angle departs therefrom.

We conclude, therefore, that the momentum of the electron in its orbit within the atom does not affect the direction of ejection in the way demanded by the theory of Auger and Perrin. While this does not constitute definite evidence against the physical reality of electronic orbits, it is a serious difficulty for the conception.

A detailed discussion of the general question here raised in the light of new experimental results will be published elsewhere.

[1] Auger, P., *J. Physique Rad.*, **8**, 1927 (85–92).
[2] Loughridge, D. H., in press.
[3] Bothe, W., *Zs. Physik*, **26**, 1924 (59–73).
[4] Auger, P., and Perrin, F., *J. Physique Rad.*, **8**, 1927 (93–111).
[5] Bothe, W., *Zs. Physik*, **26**, 1924 (74–84).
[6] Watson, E. C., in press.

THE CHARACTER OF THE GENERAL, OR CONTINUOUS SPECTRUM RADIATION

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The impacts of electrons against atoms produce two different kinds of radiation—the general, or continuous spectrum and the line spectra.

By far the greatest amount of energy radiated belongs to the general, or continuous spectrum and not to the line spectra. In the X-ray region, in particular, the continuous, or general radiation spectrum carries most of the radiated energy. In spite of this, and because the line spectra have an important bearing on atomic structure, the general radiation has received much less attention than the line spectra in the recent development of fundamental ideas in physical science. The experiments described in this note deal with the general radiation and not the line spectra.

Some time before the appearance of X-ray spectrometers in research work carried on in this country the writer performed a number of experiments on the general X-radiation. The object of these experiments was to determine whether the impacts of electrons, all of which had practically the same velocity when they struck the target of an X-ray tube, would produce homogeneous, monochromatic radiation. The experiments showed that the general radiation coming from a tube operated at a constant voltage was by no means homogeneous. The advent of the Bragg's X-ray spectrometer enabled us to show the now well-known fact

that the general radiation has a sharply defined lower wave-length limit, and that the spectrum extends from this limit toward longer wave-lengths, with a maximum intensity the exact position of which depends on experimental conditions. The average or "effective" wave-length of the continuous spectrum may be 30, 50 or even several hundred per cent greater than the short wave-length limit.

The application of the quantum theory predicts that there can be no radiation produced of higher frequency than that given by the Einstein quantum equation; for the maximum energy that an electron can have

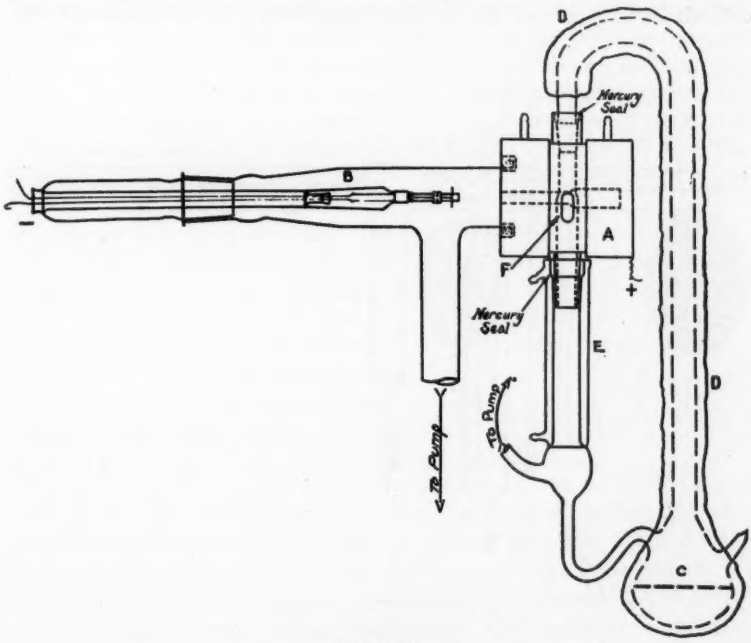

FIGURE 1

when it impinges against an atom is Ve, and if it produces radiation, the $h\nu$ value of the radiation cannot be greater than Ve. The explanation of the radiation that has wave-lengths longer than the limit has been the object of some theoretical speculations. In these the obvious assumption has usually been made that radiation of longer wave-length than the limit comes largely from the impacts against the atoms of the target of electrons that have had their energy reduced by previous collisions with other atoms. The following two questions, however, do not appear to have been definitely answered by direct experimentation. Firstly, the question as to whether when an electron hits an atom in such a way

as to produce radiation other than that of the atom's line spectrum it produces a monochromatic ray or not. Secondly, the question as to whether when an electron produces general radiation it gives up its entire kinetic energy to that radiation. The object of the researches to be described in this note has been to obtain experimental evidence bearing on these fundamental questions. In particular, the problem has been to produce and examine radiation coming from the impacts against atoms of electrons all of which have substantially the same kinetic energy at the moments of impact. To realize the conditions, electrons from a hot wire cathode have been shot into mercury vapor at a very low pressure. The

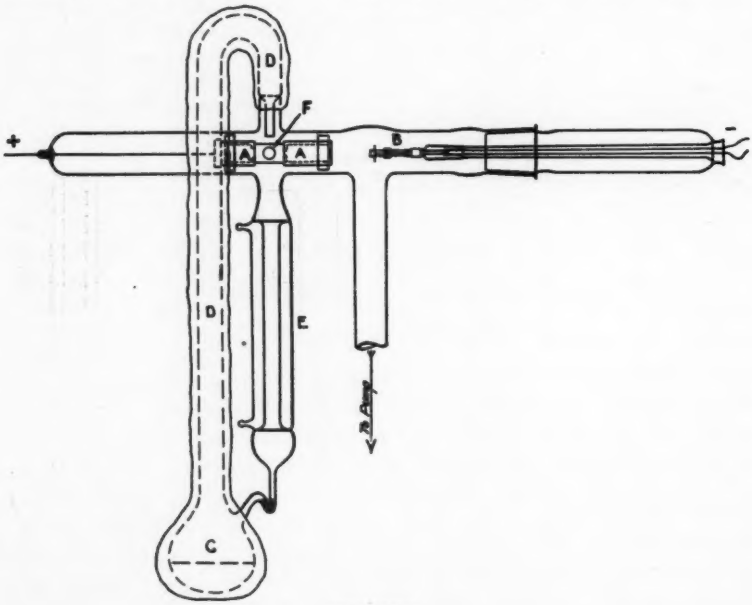

FIGURE 2

difference of potential through which the electrons fell came from a high-tension, large-capacity storage battery, the positive pole of which was put to earth and the mercury vapor was inside of a metal anode also connected to earth. Under these conditions all of the electrons must have had substantially the same kinetic energy when they entered the anode through a hole and struck the mercury atoms. The mercury vapor being at a very low pressure, probably very few of the electrons struck other atoms before they hit the atoms in such a way as to produce radiation.

Two different types of apparatus were used in the experiments. In that represented by figure 1, the anode was water cooled, and in that repre-

sented by figure 2, the anode was not cooled and got so hot during the bombardment by the electrons that no mercury could have condensed inside of it. In each figure, A represents the anode and B the Coolidge hot wire cathode which furnished the electrons. Both parts of the anode, A, in figure 2, that to the right and that to the left of the window, F, consisted of brass tubing with brass ends containing holes for the electrons to enter by. The mercury vapor came from a heated reservoir, C, and passed around through the connecting tube, D, and down through the anode. The water cooling jacket, E, condensed the mercury vapor below the anode. The mercury system, therefore, formed a kind of mercury diffusion pump. The heat to vaporize the mercury came from an electrical heating coil immersed in a thick coating of asbestos and wound around the reservoir, C, and the tube, D. In order to prevent the condensation of mercury in the connecting tube, D, the coils of wire were so arranged as to produce a much higher temperature in the tube, D, than in the reservoir, C. This heating system furnished a reasonably constant stream of mercury vapor that could be regulated by varying the electrical heating current. The mercury itself formed a sort of thin target that continually reproduced itself. The radiation generated at the points of impact of the electrons against the mercury atoms was observed through thin windows, F, made of mica in the apparatus of figure 1 and of glass in the apparatus of figure 2.

FIGURE 3

In order to determine the exact points from which the radiation passing through the windows actually came, pin-hole camera photographs were taken of the interior of the anodes. A sheet of lead with a pin hole in it was placed opposite and a short distance in front of the window, F, and several cm. back of the lead, a photographic film contained in an envelope of black paper so opaque that nothing but X-radiation could have made a photographic impression on it. Figure 3 A and B are copies of photographs taken with the apparatus of figure 1. Photograph A represents the points in the interior of the anode from which the radiation came when electrons entered the anode, but when no mercury vapor passed through the mercury system. This photograph shows the walls of the opening through which the radiation was observed, together with a drop of mercury that had collected at the bottom of the hole. Photograph B represents the source of radiation when the mercury stream passed through the anode as well as the stream of electrons. It indicates radiation from the walls of the hole and from the globule of mercury and, in addition, radiation coming from the space in which the

electrons struck the mercury vapor. It is this latter radiation which has been investigated in the experiments described below. The radiation to be examined may be separated from the wall radiation by placing a lead screen in the position occupied by the photographic film and drilling a hole through this screen at the place corresponding to the darkening on the film due to the mercury radiation. Experiments showed that no perceptible radiation came from the thin windows themselves.

The homogeneity or lack of homogeneity of a beam of radiation may be determined in different ways. Firstly, the absorption of the radiation may be measured and the effective wave-length of the beam calculated. The effective wave-length of a beam in a given case may be defined to be the wave-length of a homogeneous beam that is absorbed to the same extent as the actual beam. Secondly, the homogeneity may be examined by deflecting a beam of rays by means of prisms, or gratings, etc. (in the X-ray field by crystal gratings), and the wave-length determined by the usual procedure. The latter method is far more accurate than the former and gives in addition to the average wave-length at least a rough estimate of the distribution of energy in the spectrum. The former method, however, presents certain advantages, especially in the problem now under discussion, for far less intense radiation can be tested by it. In these experiments an important point was to use as little mercury vapor as possible in order to avoid multiple impacts of electrons against the mercury atoms. For this reason the absorption method of examining the homogeneity of the radiation has been used first. The mercury heating current was regulated so as to vaporize just enough mercury to give radiation that could be detected and measured by its ionizing effect. Radiation of such small intensity cannot be examined by means of a spectrometer.

In order to measure the absorption of the beam of rays coming from the mercury vapor an ionization chamber with a thin mica window in it and containing methyliodide was set up opposite the window, F, and lead plates with holes in them were placed in the line of the beam so that only the radiation coming from the impacts of the electrons against the mercury entered the chamber with sufficient intensity to be detected. That this was the case in the actual experiments is indicated by the fact that no perceptible ionization current could be observed when the mercury pump was not running. A quadrant electrometer measured the ionization current.

In order to examine general radiation that does not contain any of the line spectra it is necessary to operate the tube at a voltage insufficient to produce the L series lines of mercury. The longest lines in the L series of mercury require a voltage of 12,300 volts and, therefore, in the experiments here described a little less than 12,000 volts was used. A current was sent from the high-tension storage battery through a circuit containing

manganin wire resistances totalling a little more than 6,000,000 ohms. The X-ray tube was placed in parallel with 4,029,000 ohms of this resistance. This arrangement of circuits avoided the blowing out of the fuses in the battery, due to heavy discharges through the tube that occasionally occurred. By measuring the current passing through the 4,029,000 ohms with a standardized milliammeter an accurate estimate of the voltage applied to the tube could be made. The current actually used amounted to 2.95 milliamperes and the voltage actually applied to the tube was, therefore, 11,890 volts. This voltage produces the M series lines of mercury, but they have such long wave-lengths and are so easily absorbed by the windows in the X-ray tubes that no appreciable radiation from the M series could get into the ionization chamber. Thus the ionization current must have been due entirely to general radiation.

The short wave-length limit of the X-radiation produced by the above difference of potential of 11,890 volts was 1.040 Å. The point to be decided by these experiments was how close to this particular wave-length limit the average, or effective wave-length of the beam of X-rays coming from the mercury vapor really lay.

Absorption measurements with both types of tubes (Fig. 1 and Fig. 2) have been made with substantially the same results in each case. An experiment with the hot anode tube represented by figure 2 will be described in detail. In this experiment the fraction of radiation passing through a thin sheet of aluminum was determined. The sheet had an area of 9.801 cm.^{2} and it weighed 0.2718 gram. Hence the mass per unit area, ρd, was 0.02773. The average value of the mass coefficient of absorption μ/ρ for aluminum given by Richtmyer and Allen is 13.9 for $\lambda = 1.00$ and 20.0 for $\lambda = 1.10$. In order to obtain an independent measurement of the absorption of X-rays of given wave-lengths by *this particular* sheet of aluminum, and in order to correct for any possible lengths by the sheet of aluminum was determined. The γ_1 and γ_4 lines in the L series of tungsten were employed with wave-lengths of 1.096 and 1.026 Å, respectively. An X-ray spectrometer with a tungsten target tube was adjusted so as to reflect each of these lines successively and the fraction of each passing through the sheet of aluminum was measured. In the case of the γ_1 line, $\lambda = 1.096$ Å, the ratio of the intensity of radiation falling on the aluminum to that coming through it amounted to 1.747. This gives a value of $\mu/\rho = 20.2$, in close agreement with the above-mentioned measurements of Richtmyer and Allen. In the case of the γ_4 line the ratio of intensities was 1.544 giving a value of $\mu/\rho = 15.9$. Dr. E. Lorenz very kindly checked these values in separate experiments.

In order to determine the effective wave-length of the radiation coming from the mercury through the window of the X-ray tube the sheet of aluminum, calibrated as described, was placed between the tube and

the ionization chamber, the distance from the sheet of aluminum to the window of the ionization chamber being large so that the amount of scattered radiation from the sheet of aluminum that entered the ionization chamber might be neglected. Two series of measurements were made of the ratio of the ionization currents without and with the sheet of aluminum in place. The average value of the ratio of the radiation falling upon the aluminum sheet to that which pass through was 1.74 in the first series and 1.76 in the second. These ratios are probably correct to within one or two per cent. They are the same as that obtained in the calibration of the sheet of aluminum with the γ_1 line within the limits of error of the measurements. The effective wave-length, therefore, of the X-ray beam coming from the mercury vapor through the window under the conditions of voltage, etc., described above does not differ perceptibly from that of the γ_1 line in the L series of tungsten, namely: $\lambda = 1.096$ Å.

Another series of similar measurements was made with an additional sheet of aluminum inserted as a filter between the X-ray tube and the ionization chamber. Under these conditions, the ratio of the radiation falling on the calibrated sheet of aluminum to that which passed through the calibrated sheet was 1.79, a slightly greater value than that obtained without the additional sheet of aluminum. The difference between the two, however, is scarcely greater than the errors of measurements in experiments of this kind. The ratio 1.79 corresponds to an effective wave-length of 1.11 Å. From these experiments, we may conclude that the aluminum filter does not reduce the effective wave-length coming from the mercury perceptibly and that this effective wave-length is 1.10 Å to within the limits of error of the measurements.

Since the short wave-length limit of the spectrum for the voltage used was 1.04 Å, we see that the effective wave-length of the beam was only about 6% greater than the limit. This difference, although small, is greater than the probable error of measurement and indicates that the beam of X-rays, although approximately homogeneous, was not exactly so. It was a great deal nearer homogeneity than would have been the case had the target of the X-ray tube been a solid one.

The effective wave-length, λ_e, of a beam of X-rays may be defined by the following equation:

$$e^{-f(\lambda_e)} \int I d\lambda = \int I e^{-f(\lambda)} d\lambda,$$

where $f(\lambda)$ is nearly proportioned to λ^3.

If we calculate the effective wave-length of a band in the spectrum of uniform intensity, I, extending from $\lambda = 1.04$ to $\lambda = 1.16$, in other words, a band about 10% in width, we find that its value is $\lambda_e = 1.10$. Hence, the beam of X-rays actually coming from the mercury through the window

in our experiments is equivalent, insofar as its absorption in the sheet of aluminum is concerned, to a band 10% broad extending from the short wave-length limit of the radiation toward longer wave-lengths.

It does not seem probable, however, that the actual X-ray band is of uniform intensity throughout. One would naturally expect the form of the intensity wave-length curve to be similar to that of a solid target, although very much narrower. The intensity curve for a solid target rises rapidly from the short wave-length limit to a maximum and then falls with decreasing rapidity toward the longer wave-lengths. If this is the actual shape of the intensity curve for the mercury radiation and if much of the radiation has wave-lengths longer than 1.16, most of the X-ray quanta coming from the mercury through the window must have wave-lengths lying between the short wave-length limit 1.04 and 1.10 Å. In any case, we see that a large proportion of the electrons give up most of the kinetic energy that they ever possessed to the X-ray quanta when they produce the X-radiation.

There are several reasons why the electrons that produce the X-radiation when they hit the mercury atoms may not all have the maximum kinetic energy represented by the voltage applied to the tube, namely Ve. If electrons come from a Coolidge cathode held at a constant potential below the anode, there is a slight spread of velocities when they pass through a hole in that anode. Some experiments made by Mr. Hudson in our laboratory on the magnetic deflection of such electrons after they had passed through a long tube indicated that the spread of velocity was not large for voltages above 10,000 volts. The kinetic energy of only a few of the electrons differed from the value Ve after they had come through the tube. In the experiments described in this note on the impacts of electrons against mercury atoms, some of the electrons may have hit mercury atoms before the impacts which actually produced the X-radiation and may have had their kinetic energies reduced by ionization. Again the impacts of the electrons against the solid interior walls of our anodes must have generated a large amount of X-radiation and this X-radiation must have produced photo-electrons with kinetic energies less than the value of Ve. The X-radiation produced by the impacts of the electrons against the solid walls of our anodes did not enter the ionization chamber, for no ionization could be detected when the mercury pump was not running. Some of the photo-electrons from this radiation, however, must have struck the mercury vapor atoms and these impacts were impacts of electrons of smaller kinetic energy than Ve. It is impossible to estimate what proportion of the electrons hitting the mercury atoms had kinetic energies less than Ve. There must have been some such collisions, however, and this fact means that on the average the amount of energy transformed from the kinetic energy of an electron to the energy of the

quantum must be closer to the kinetic energy of the electron *at the time of impact* than to Ve. In other words, in a large number of impacts, at least, the electron transfers almost if not all of its kinetic energy to the quantum of radiation when it produces that quantum, and the radiation thus produced is nearly if not exactly monochromatic.

THE STRUCTURE OF THE ATMOSPHERIC ABSORPTION BANDS OF OXYGEN

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Communicated July 18, 1927

Although the atmospheric oxygen bands have been the subject of many investigations, the details of their structure have remained only partially explained. The earlier investigators[2] did not have any theory to guide them and it can readily be understood that they often came to contradictory results. In the present paper we have tried to account for the structure of the bands as completely as possible. Our progress has been based on new wave-length measurements which we believe to be more accurate than the previous ones.

1. A recent paper by one of us,[3] gives details of the derivation with the interferometer of the wave-lengths of the lines in the α, B and A bands best suited to that method. With these lines as standards, the other lines in these bands have been measured on spectrograms obtained with a large Michelson plane grating used in the third and fourth orders at 30-feet focus. Such photographs were made at both high and low altitudes of the sun, since the faintest atmospheric lines are best observed with maximum air-path and the strongest ones with minimum air-path. At best the measurement of the widest lines is difficult and of relatively low accuracy. For example, some lines in the A band have a total width of about $1A$, even when observed on Mount Wilson with solar altitude exceeding 45°. The finer lines, on the other hand, can be seen only with high resolving power, and even with the sun close to the horizon they are faint and hard to measure. For the widest lines in the A band the usual micrometer measurements were supplemented by the use of a registering, thermoelectric microphotometer, giving a definite increase of accuracy. Some of these lines give evidence that under more favorable conditions of observation they might be resolved into close pairs.

For the α' band, $\lambda\,5788 - \lambda\,5834$, we have made no measurements, but have derived the wave-lengths on the International system by

subtracting 0.216A from the values given in Rowland's Table of Solar Wave-Lengths. The main part of this correction was obtained from the revision of the Rowland Table now in progress, with a slight alteration to allow for the difference between the standards of iron and those of neon. Throughout this paper all wave-lengths are expressed on the neon scale. Some small differences between wave-lengths stated here and those given in the work of Babcock to which reference has been made are accounted for by additional observations made specially for our purpose. The discordance between the values of Meggers[4] and those given below is a natural result of the methods and approximations which he employed. The numerical data relating to the bands appear in table 1. The band designated A' is described later.

TABLE 1

A BAND

	λ	I	$\nu_{VAC.}$	λ	I	$\nu_{VAC.}$
	$P_1(J)$				$P_2(J)$	
0				7620.996	15	13,118.04
2	7624.493	23	13,112.02	23.289	22	14.11
4	28.203	25	05.64	27.050	24	07.63
6	32.150	25	13,098.87	31.021	24	00.81
8	36.317	25	91.72	35.196	24	13,093.64
10	40.694	23	84.22	39.583	22	86.12
12	45.297	19	76.34	44.197	18	78.22
14	50.120	16	68.10	49.032	15	69.96
16	55.172	13	59.47	54.078	12	61.34
18	60.447	10	50.48	59.359	9	52.34
20	65.938	6	41.13*a*	64.873	6	42.95*S*
22	71.664	5	31.40*a*	70.598	5	33.21*a*
24	77.617	4	21.29*a*	76.561	4	23.09*a*
26	83.798	3	10.82*a*	82.756	3	12.59*a*
28	90.215	2	12,999.37	89.175	2	01.72*a*
30	96.866	0	88.73*a*	95.836	0	12,990.47*a*
32	7703.756	0	77.11*a*	7702.733	0	78.83*a*
		$R_1(J)$			$R_2(J)$	
2	7616.143	10	13,126.40	7615.048	17	13,128.28
4	13.180	19	31.51	12.054	23	33.45
6	10.450	23	36.21	09.290	27	38.22
8	07.911	24	40.60	06.748	27	42.61
10	05.628	24	44.54	04.431	24	46.61
12	03.553	17	48.13	02.346	18	50.22
14	01.695	17	51.35	00.45	15	53.50
16	00.09	15	54.12	7598.760	24	56.43*b*
18	7598.760	24	56.43*b*	97.435	16	58.72*b*
20	97.435	16	58.72*b*	86.208	10	60.85
22	96.479	8	60.38	95.245	10*d*	62.51*b*
24	95.759	5	61.62	94.500	6	63.81*a*
26	95.245	10*d*	62.51*b*	93.988	3	64.70*a*
28	94.970	4	62.99*b*	93.695	3	65.20*b*,*a*
30	94.970	4	62.99*b*	93.695	3	65.20*b*, *a*

TABLE 1—*Continued*

A' BAND

j	λ	I $P_1'(j)$	$\nu_{VAC.}$	λ	I $P_2'(j)$	$\nu_{VAC.}$
0	.....	..	.....	7619.691	00	13,120.29
2	7622.998	0	13,114.59	21.818	0	16.62
4	26.513	0	08.55	25.345	0	10.56
6	30.232	0	02.16	29.084	0	04.13
8	34.161	0	13,095.42	33.028	0	13,097.36
10	38.298	0	88.32	37.172	0	90.25
12	42.639	0	80.89	41.53	0	82.79
14	47.188	0	73.11	46.10	0	74.97
16	51.95	0	64.90	50.88	0*d*	66.80
		$R_1'(j)$			$R_2'(j)$	
4	.....	..	.....	7611.002	0	13,134.94
6	7609.710	0	13,137.49	08.552	1*d*	39.49
8	07.358	1	41.55	06.211	1*d*	43.54
10	05.194	00	45.29	04.004	0	47.35
12	03.212	0	48.72	02.028	00	50.77

B BAND

j	λ	I $P_1(j)$	$\nu_{VAC.}$	λ	I $P_2(j)$	$\nu_{VAC.}$
0	.....	..	.....	6883.832	10	14,522.792*a*
2	6886.743	12	14,516.656*a*	85.754	11	18.740*a*
4	89.903	14	09.997*a*	88.948	13	12.008*a*
6	93.309	15	02.828*a*	92.369	14	04.804
8	96.965	15	14,495.140*a*	96.037	14	14,497.089*a*
10	6900.868	15	86.941*a*	99.954	14	88.860*a*
12	05.023	14	78.224*a*	6904.117	14	80.123*a*
14	09.431	13	68.988*a*	08.534	13	70.865*a*
16	14.090	11	59.235*a*	13.200	11	61.099*a*
18	19.002	9	48.972*a*	18.122	9	50.810*a*
20	24.172	9	38.183*a*(*S*)	23.302	9	39.998*a*
22	29.598	4	26.88	28.728	4	28.69
24	35.283	2	15.05	34.421	2	16.84
26	41.231	1	02.70	40.381	1	04.46
28	47.45	0	89.81	46.614	0	14,391.54
		$R_1(j)$			$R_2(j)$	
2	6879.928	6	14,531.035*a*	6879.041	12	14,532.907*a*
4	77.637	12	35.874*a*	76.715	13	37.824*a*
6	75.590	13	40.202*a*	74.653	13	42.183*a*
8	73.798	12	43.992*a*	72.843	12	46.013*a*
10	72.247	11	47.266*a*	71.285	10	49.311*a*
12	70.946	8	50.029*a*	69.942	7	52.16*b*
14	69.942	7	52.16*b*	68.915	7	54.33*b*
16	69.093	6	53.953*a*	68.106	6	56.05
18	68.553	3	55.10*b*	67.554	5	57.22
20	68.237	6*d*	55.77*b*	67.247	6	57.87
22	68.237	6*d*	55.77*b*	67.209		57.95
24	68.429	1	55.36	67.403	1	57.54
26	68.915	7	54.33*b*	67.865	0	56.56
28	69.626	00	52.83(*S*)	68.553	3	55.10*b*

TABLE 1—*Continued*

α BAND

j	λ	I $P_1(j)$	$\nu_{VAC.}$	λ	I $P_2(j)$	$\nu_{VAC.}$
0	.....	..	.....	6287.747	1	15,899.56*a*
2	6289.397	1	15,893.30*a*	90.222	2	95.39*a*
4	92.162	2	86.41*a*	92.957	3	88.40*a*
6	95.178	3	78.81*a*	95.961	3	80.79*a*
8	98.456	2	70.58	99.229	3	72.53
10	6302.001	2	61.78*a*	6302.764	2	63.60
12	05.810	2	52.12*a*	06.566	2	54.02
14	09.885	2	41.89	10.638	1	43.78
16	14.234	0	31.00	14.977	0	32.86
18	18.851	00	19.45	19.587	00	21.30
		$R_1(j)$			$R_2(j)$	
2	6284.537	0	15,907.68	6283.795	1	15,909.56*a*
4	82.725	2	12.27	81.956	2	14.22*a*
6	81.178	1	16.19*a*	80.393	2	18.18*a*
8	79.896	2	19.44*a*	79.100	3	21.46*a*
10	78.878	2	22.02*a*	78.078	2	24.05
12	78.126	2	23.93	77.311	2	25.99
14	77.643	1	25.15	76.818	1	27.24*a*
16	77.425	0	25.70	76.609	2*d*	27.77*a,b*
18	77.533	000	25.43	76.609	2*d*	27.77*b*

α′ BAND

j	λ	I $P_1(j)$	$\nu_{VAC.}$	λ	I $P_2(j)$	$\nu_{VAC.}$
0	.....	..	.....	5796.088	0	17,248.24*S*
2	5798.182	4	17,242.01*S*	97.499	00	44.05
4	5800.628	0	34.74(*S*)	99.969	00	36.70
6	03.333	0	26.71	5802.669	0	28.69
8	06.294	0	17.92	05.624	0	19.91
10	09.525	00	08.35	08.869	00	10.30
12	13.054	00	17,197.90(*S*)	12.400	00	99.84(*S*)
14	16.841	00	86.71(*S*)	16.265	00	17,188.41(*S*)
16	20.894	000	74.74	20.294	000	76.51
18	25.295	000	61.77(*S*)	24.642	000	63.69
20	30.089	000	47.66(*S*)	29.317	0000*N*	49.93(*S*)
22	34.854	0000*N*	33.65(*S*)	34.224	0000	35.50(*S*)
		$R_1(j)$			$R_2(j)$	
2	5793.394	000	17,256.27(*S*)	5792.768	00	17,258.13
4	91.924	00	60.64	91.269	00	62.60
6	90.769	00	64.09	90.097	0	66.09
8	89.855	0	66.81	89.202	0*d*	68.76*b*
10	89.202	0*d*	68.76*b*	88.539	0	70.74
12	88.879	00	69.72*b*	88.182	00	71.80*b*
14	88.879	00	69.72*b*	88.089	000	72.08
16	88.994	000	69.38	88.182	00	71.80*b*
18	89.484	000	67.92	88.774	00	70.04

2. *The Empirical Structure of the Bands.*—We number the lines (table 1) in such a way that the first complete doublet on both sides of the zero

line has the number two. The other lines will then have the successive even numbers, as the doublets belonging to the odd numbers are missing. We designate the longer wave-length component by a subscript one and the other component by a subscript two. Table 2 shows that the differences, $R_i(j) - P_i(j)$ are identical for the four bands under consideration,* ($i = 1, 2$), which means that $R_i(j)$ and $P_i(j)$ have the same initial† state.

TABLE 2

$R_i(j) - P_i(j)$

		A	B	α	α'	A'
2	$\begin{cases} \\ \end{cases}$	14.38 14.17	14.379 14.167	14.38 14.17	*14.26 14.08	
4	$\begin{cases} \\ \end{cases}$	25.87 25.82	25.877 25.816	25.86 25.82	25.90 25.90	 24.38
6	$\begin{cases} \\ \end{cases}$	37.34 37.41	37.374 37.379	37.38 37.39	37.38 37.40	35.33 35.36
8	$\begin{cases} \\ \end{cases}$	48.88 48.97	48.852 48.924	48.86 48.93	48.89 *48.85	46.13 46.18
10	$\begin{cases} \\ \end{cases}$	60.32 60.49	60.325 60.451	60.24 60.45	*60.41 60.44	56.97 57.10
12	$\begin{cases} \\ \end{cases}$	71.79 72.00	71.805 *72.04	71.81 71.97	*71.82 *71.96	67.83 67.98
14	$\begin{cases} \\ \end{cases}$	83.25 83.54	*83.17 *83.46	83.26 83.46	*83.01 83.67	
16	$\begin{cases} \\ \end{cases}$	94.65 *95.09	94.718 94.95	94.70 *94.91	94.64 *95.29	
18	$\begin{cases} \\ \end{cases}$	*105.95 *106.38	*106.03 106.41	105.98 *106.47	*106.15 106.35	
20	$\begin{cases} \\ \end{cases}$	*117.59 *117.90	*117.59 117.87			
22	$\begin{cases} \\ \end{cases}$	128.98 *129.30	*128.89 129.26			
24	$\begin{cases} \\ \end{cases}$	140.33 140.72	140.31 140.70			
26	$\begin{cases} \\ \end{cases}$	*151.69 152.11	*151.63 152.10			
28	$\begin{cases} \\ \end{cases}$	*163.02 *163.48	163.02 *163.56			

We can, therefore, write

$$\left.\begin{aligned} P_i(j) &= F_i'(j) - F_i''(j+1) \\ R_i(j) &= F_i'(j) - F_i''(j-1) \end{aligned}\right\} \quad (1)$$

The dependence on the final state cannot be tested by the combination principle, as we have no two bands with the same initial vibrational state. But the form (1) is the simplest assumption and makes the structure of the bands analogous to that of other well-known bands.

The lines can be represented approximately by a quadratic formula, though there are small but distinctive deviations in the P_2 and R_2 branch for small values of j. If we neglect these at first, we may write

$$F(j) = A + Bm^2 - \beta m^4 \quad (2)$$

in which A represents that part of the energy (divided by h) which is not dependent on the rotational quantum number; β is a small correction term which is due to the non-rigidity of the molecule. We want to calculate the constants which are involved in forming the bands. On account of (1) and (2), the differences tabulated in table 2 are equal to

$$R_i(j) - P_i(j) = F_i''(j+1) - F_i''(j-1) = 4B_i''m_i - (8m_i^3 + 8m_i^3)\beta_i''$$

and can be used to calculate the values for B_i'', β_i'' and m_i for the final state. In the same way, the constants for the initial state can be obtained from

$$R_i(j+1) - P_i(j-1) = F_i'(j+1) - F_i'(j-1) = 4B_i'm_i - (8m_i^3 + 8m_i^3)\beta_i'.$$

Only the first ten lines (to R (18) and P (16) incl.) were used for the calculation of the constants in order to be sure that no higher terms have an appreciable influence, and because the wave-lengths of overlapping lines in the head, especially of the A band, cannot be determined with great accuracy. The values for the constants are given in table 3: n means the vibrational quantum number; the value for the zero line ν_{01} of each band is given in the row of its initial state. Not much significance can be attached to the values for the correction term β. The values of m calculated in this way are, for the first few terms:

final state				
	m_1 : 2.500	4.500	6.500	8.499
	m_2 : 2.448	4.462	6.472	8.459
initial state (B band)				
	m_1 : 1.502	3.501	5.504	7.502
	m_2 :	3.448	5.454	7.452

It is seen from this that the m-values for the F_1' and F_1'' state are very nearly half integers, whereas the values for the F_2' and F_2'' state are considerably lower. On the whole, the F_2 terms cannot be so well represented

by a formula of the type (1), and there are distinct deviations for small values of the rotational quantum number. The significance of this will be discussed later on. The zero line for the P_2 and R_2 branch is approximately 1.9 cm.$^{-1}$ higher than that of the P_1 and R_1 branch.

TABLE 3

	B_1	B_2	β_1	β_2	ν_{01}
Initial state					
A, $n = 0$	1.390	1.402			13,122.97
B, $n = 1$	1.372	1.384	$5.75 \cdot 10^{-6}$	$10.63 \cdot 10^{-6}$	14,525.72
α, $n = 2$	1.354	1.364			15,902.50
α', $n = 3$	1.336	1.346			17,251.19
A', $n = 1$	1.311	1.318	———	———	13,123.00
Final state					
$n = 0$	1.438	1.447	$6.31 \cdot 10^{-6}$	$7.92 \cdot 10^{-6}$	
A', $n = 0$	1.354	1.361	———	———	

The observed doublet separation, which is the same for all bands, can be accounted for with the above terms. If we express the deviation of m from half integer values in F_2 by a linear term αm, the doublet separation becomes

$$\Delta P(m) = \nu_{02} - \nu_{01} - B_2'' + B_1'' - \alpha'' - 2(B_2'' - B_1'')m + (\alpha' - \alpha'')m + (C_2 - C_1)m^2$$

$$\Delta R(m) = \nu_{02} - \nu_{01} - B_2'' + B_1'' + \alpha'' + 2(B_2'' - B_1'')m + (\alpha' - \alpha'')m + (C_2 - C_1)m^2$$

$$(C = B' - B'').$$

These formulæ give all the properties of the observed doublet separation, although an *exact* quantitative test cannot be obtained, because the small differences $B_2 - B_1$, $\alpha' - \alpha''$, etc., are not known with sufficient accuracy. The value for the doublet separation of the zero lines is found to be 1.93. The zero lines follow the formula

$$\nu_{01} = 13122.965 + 1415.017n - 11.911n^2 - 0.3525n^3 \quad (n = 0,1,2,3,..)$$

in which the constants differ only slightly from those calculated by Kratzer[5] from the old data. It is also seen from table 3 that the constants B' have the linear dependence on the vibrational quantum number postulated by Kratzer's theory.

The A band includes a number of *weak lines*, and those in the tail are at once seen to form doublets of the same kind as the strong lines. In the head also doublets can be picked out, and then the constants can be calculated in the same way as for the strong branches. These are also given in table 3. The weak lines have been called A'. It appears that the weak branches have exactly the same structure as the strong ones. The only difference is the value for the constant B. The zero lines for the strong and weak branches are the same within the limits of experimental error. The weak branches are too faint to be observed in the other bands. There is still a small number of unclassified faint lines of atmos-

pheric origin, but it is not certain whether they belong to these bands. They may be partly due to water vapor.

The significance of the empirical structure of the bands for the structure of the oxygen molecule is not yet satisfactorily cleared up in all points. Approximately we have $B = h/8\pi^2 cJ$, and from this it follows that the moment of inertia J for the O_2-molecule in its normal state is $19.27 \cdot 10^{-40}$, and in the excited state, $19.93 \cdot 10^{-40}$, which gives for the distance between the nuclei $1.205 \cdot 10^{-8}$ and $1.225 \cdot 10^{-8}$ cm., respectively.

3. Mulliken[6] has developed a systematic analysis of band spectra to which Hund[7] gave a theoretical background. According to this theory the type of an electronic molecular term is determined by quantum numbers which are analogous to those used for the classification of line spectra. The resultant spin of the electrons, s, determines the multiplicity of the terms.

The components of the electronic angular momentum are σ_k and σ_s, and the spin along the nuclear axis is $\sigma = \sigma_k + \sigma_s$. If $\sigma_k = 0, 1, 2, \ldots$, we have $S, P, D, \ldots$ terms. For all particulars it is necessary to consult the above-mentioned papers.

The fact that both initial and final state of the atmospheric absorption bands are not single for zero rotation shows that neither can be a singlet nor an S state. It seems very probable that we have here a $^3D^3P$ transition. For a ^3P state $s = 1$, $\sigma_k = 1$, $\sigma = 0, 1, 2$; for a ^3D state $s = 1$, $\sigma_k = 2$, $\sigma = 1, 2, 3$. Hund derives a selection rule $\Delta\sigma = 0$, if $\sigma_k > 0$, which is fulfilled, e.g., in the second positive nitrogen group. On account of this rule, only the levels with $\sigma = 1$ and 2 can combine with each other. The fact that alternate rotational levels are suppressed because we have a symmetrical molecule eliminates the Q branches. There are, however, some difficulties in accounting for the missing lines. We confine ourselves to these few remarks about the electronic structure, as this will be discussed more fully, together with the electronic structure of the Runge bands, in a subsequent paper.

We are indebted to Dr. St. John for unpublished data on the revision of Rowland's Table of Solar Spectrum Wave-Lengths, to Mr. W. P. Hoge for assistance in some of the measurements, and to Dr. Pettit and Miss Ware for operating the microphotometer for us.

REMARKS ON THE TABLES

Intensities in A band from Meggers; in B, α, α', from Rowland.
S indicates a line influenced by a solar line.
(S) is used where such influence is possible, but doubtful.
a indicates lines for which wave-lengths have highest weight.
b indicates lines occurring in more than one place.
* in table 2 means that one of the lines involved is a b or an S line.

REFERENCES

* j means here only the ordinal number of the lines.

† The expressions initial and final state are always used from the point of *emission*.

[1] NATIONAL RESEARCH FELLOW.

[2] O. C. Lester, *Astrophysical Journal*, **20**, 81, 1904. This paper contains the references to the preceding work. R. Fortrat, *Ann. d. Phys.*, **3**, 384, 1915; T. Heurlinger, *Untersuchungen über die Bandenspektra*.

[3] H. D. Babcock, *Astrophysical Journal*, **65**, 140, 1927; Mt. Wilson Contr., No. 328.

[4] W. F. Meggers, *Publications Allegheny Observ.*, **6**, No. **3**, 1919.

[5] A. Kratzer, *Ann. d. Phys.*, **68**, 122, 1922.

[6] R. S. Mulliken, *Phys. Rev.*, **26**, 561, 1925; *Proc. Nat. Acad. Sci.*, **12**, 144 and 151, 1926; *Phys. Rev.*, **28**, 481 and 1202, 1926; also R. T. Birge, *Nature*, **117**, 300, 1926.

[7] F. Hund, *Zs. f. Phys.*, **26**, 657, 1926; **40**, 742, 1927; **42**, 93, 1927.

REVISION OF ROWLAND'S PRELIMINARY TABLES OF SOLAR SPECTRUM WAVE-LENGTHS

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The first extensive list of wave-lengths in the solar spectrum was made by A. J. Ångström in 1868.[1] In recognition of this work his name has been given to the unit of wave-length in the international system.[2] In 1893 Rowland published "A New Table of Standard Wave-Lengths" containing several hundred lines in the solar spectrum.[3] This represented the results of some ten or more years of investigation, during which period he constructed the ruling engine and produced the first concave grating. This table of wave-lengths so surpassed in accuracy all previous work that it received world-wide acceptance and was made the basis of his "Preliminary Table of Solar Spectrum Wave-Lengths,[4]" which for a generation has been the world's standard and the vade-mecum of astronomers and physicists.

The starting point of the Rowland system was the mean wave-length of the D_1 line of sodium as referred to the standard meter by five investigators:

	λ	WGT.
Ångström	5895.81	1
Müller and Kempf	5896.25	2
Kurlbaum	5895.90	2
Peirce	5896.20	5
Bell	5896.20	10

Mean 5896.156 in air at 20°C. and 760 mm. pressure

From the sodium line about 15 lines in the visual region were determined by the method of coincidences between spectra of different orders. The list was increased by interpolation, extended into the ultra-violet by photography, and during a period of eight or nine years the thousand lines, more or less, of the "New Table of Standards" were intertwined with each other in an immense number of ways. Rowland says, in a note, that the

calculations had involved about a million figures, of which he had personally written more than half and was not anxious for more labor of that kind. To this "Table of Standards" the 20,000 lines of the "Preliminary Table of Solar Spectrum Wave-Lengths" were referred, and, by adjustments and interconnections, the whole was welded into what, for the means and methods then available, was a remarkably consistent system, but short of the degree of accuracy demanded by present-day spectroscopy.

Owing to the inadequacy of the grating for absolute measures,[5] the wave-length of the sodium line adopted by Rowland as his primary standard was too long by 0.212 Å. If this were the only error, the correction throughout his tables would be proportional to wave-length and easily applied; but the method of coincidences[6] and the adjustments for bringing the solar and arc values into agreement introduced other and irregularly distributed errors.[7] No allowance was made for the effect of the rotation and the orbital motion of the earth. In fact, Rowland was never convinced that real differences existed between solar and laboratory wave-lengths—differences upon which now rests the evidence of the gravitational displacement of solar lines predicted by Einstein.[8]

The next great advance was made in the Michelson interferometer. In 1893 Michelson and Benoit determined the absolute wave-length of the red cadmium line (in terms of the *mètre des archives*) with a hitherto undreamed-of precision.[9] Fourteen years later, Benoit, Fabry and Perot,[10] working in collaboration, redetermined this fundamental wave-length, obtaining an extraordinary agreement with the original result of Michelson and Benoit:

RED CADMIUM LINE

Reduced to dry air at 15°C. and 760 mm. pressure

1907	Benoit, Fabry, and Perot series 3	6438.4696 Å
	4	6438.4695
	5	6438.4692
	7	6438.4700
	Mean.....	6438.4696
1893	Michelson and Benoit, reduced	6438.4700

In 1907 the value 6438.4696 Å was adopted as the absolute wave-length of the red cadmium line, the Primary Standard in the international system of wave-lengths.[11] This wave-length is thought to be known to one part in ten million, that is, the absolute error referred to the *mètre des archives* is less than 0.001 Å.

With the interferometer, a spectral line can be directly referred to the primary standard with the consequent elimination of the cumulative errors inherent in a step-by-step method. This is of the utmost advantage in the fixing of a series of standard wave-lengths well distributed through the spectrum. Under the auspices of the *International Union for Co-*

operation in Solar Research a list of secondary standards was established, each the mean of three independent interferometer determinations of selected lines in the iron arc. To these were later added tertiary and auxiliary standards, interpolated between the secondary standards. The interpolations have been made by both gratings and interferometers, the adopted wave-lengths being the means of 4 to 8 closely agreeing measures.[12]

For more than a decade investigations have been carried on at the Mount Wilson Observatory for the purpose, among others, of determining solar wave-lengths in the international system. The difficulties met and the methods applied to the problems that arose need not be detailed here, as they have been referred to in various *Contributions* from Mount Wilson by Mr. Babcock, Miss Ware and myself.[13] It suffices to say that the present values rest upon two independent series of measurements, in each of which a large number of lines, well distributed in the spectrum and sufficiently isolated to serve as reliable standards, have been measured. One series depends upon simultaneous exposures to the center of the sun and the standard iron arc, made with the 30-foot spectrograph and the 60-foot tower telescope, in the earlier period, and with the 75-foot spectrograph and the 150-foot tower telescope, in the later period. The other series by Mr. Babcock was made with the interferometer, using at first the Snow telescope on Mount Wilson for the accessory dispersion, and later the laboratory installation in Pasadena.

The two series, corrected for the rotation and the orbital motion of the earth, are in excellent agreement; the differences for individual lines rarely exceed 0.002 Å, while the systematic deviation is of the order of a few ten-thousandths of an Ångström. With the means of these two series as reference points, the wave-lengths of a still larger number of lines were interpolated upon spectrograms of high dispersion. For the region λ 3560–λ 7330 the differences, Rowland *minus* international, combined in groups of five were plotted as ordinates on a scale of 0.001 Å per 2.5 mm., against the wave-lengths as abscissæ on a scale of 1 Å per 1.25 mm. From the smooth curve 4.5 m. long passed through the points, the corrections, R − I. A., were taken for lines not measured. In the region λ 2975–λ 3560, not accessible with the high-dispersion equipment, an indirect method was used:

$$\text{Rowland} - \text{Sun I. A.} = \text{correction}$$
$$\text{Sun I. A.} - \text{Arc I. A.} = \Delta\lambda$$
$$\Delta\lambda = \text{Rowland} - \text{Arc I. A.} - \text{correction.}$$

$\Delta\lambda$ was determined for the region immediately to the red of λ 3560, and used in the region to the violet, for which Rowland − Arc I. A. was found by using the I. A. arc wave-lengths of Fe, Ni, Co and Cr.

The differences, R − Sun I. A., are remarkably consistent through short

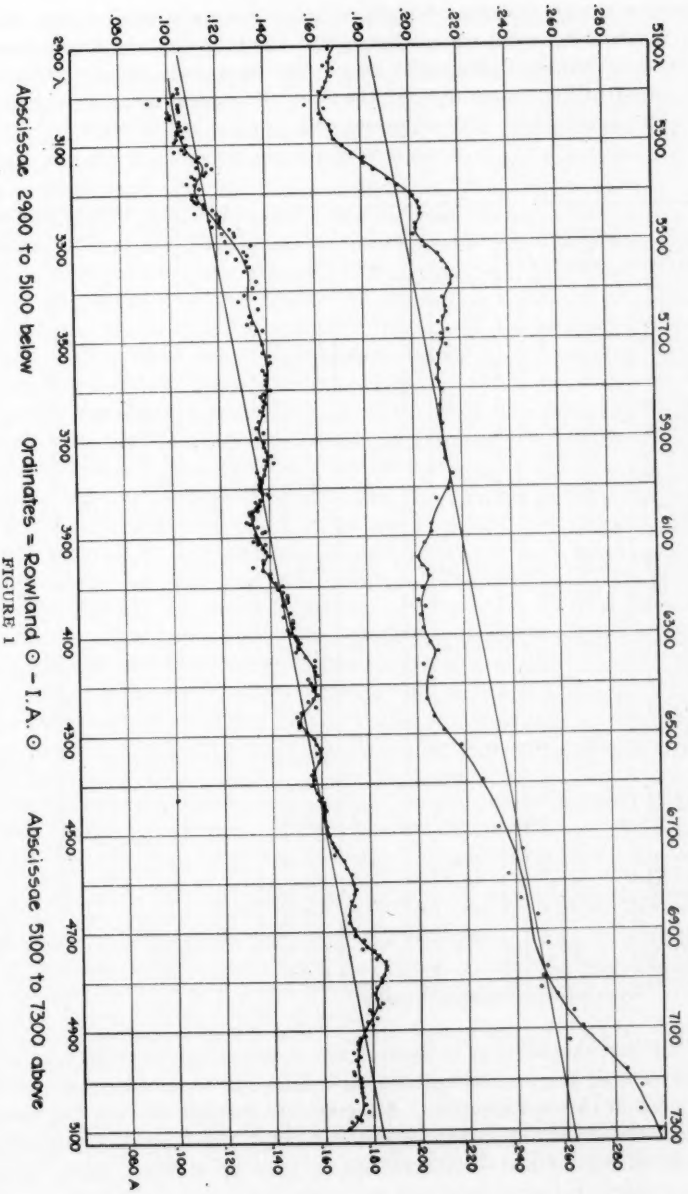
Abscissae 2900 to 5100 below
Ordinates = Rowland ⊙ − I.A. ⊙
Abscissae 5100 to 7300 above

FIGURE 1

spectral ranges, the mean deviations in the five-line groups being ± 0.002 Å; but over extended regions the corrections to the Rowland wave-lengths vary in a curiously progressive way. This appears in the deviations from the straight line drawn through the point corresponding to R − Sun I. A. for the sodium line, with ordinates equal to $\lambda \times 0.212/5900$.

REVISION OF THE SOLAR SPECTRUM TABLE

λ I A	Element	Int Disk	Int Sp	Class Temp	Class Pr	E P	λ I A	Element	Int Disk	Int Sp	Class Temp	Class Pr	E P
4212.853	CN	-1					4220.171		-1				
4212.978	Fd?	-1N				1.447	4220.349m	Fe	3	4	IV	b	
4213.166	CN,Cr	ON	1	IV	b		4220.484		-3				
4213.277	CN	-2					4220.579		-2				
4213.422		-1					4220.654	Y?	-1				
4213.522	CN	-2					4220.808		-1				
4213.655m	Fe-CN	3	3	IV	b		4221.021		-2				
4213.834	Zr	0		III?		0.601	4221.174		-1				
4213.913	-CN	0	2				4221.310		-1				
4214.041	CN	-1					4221.475		1N	1			
4214.138		-2					4221.579	Cr	0	0	IV	d	
4214.249	CN	-1					4221.695	Ni?	-1			d	3.292
4214.370	CN	-1					4221.817		-2				
4214.477		-1					4222.025		-2				
4214.630	CN	-1					4222.223m	Fe	5	7	III	d	2.439
4214.837	CN	-1					4222.453		-1d?				
4214.917	CN	-1					4222.610	Ce*	0	1			
4215.063	-CN	0	0				4222.732	Cr	0	0	III	d	3.000
4215.179	CN	-1					4222.903		-1N				
4215.301	CN	-1					4223.098		1N	2			
4215.423	CN-Fe	2	1	IV	b		4223.238		0	0			
4215.545	CN Zr*	5d?	7	II		0.000	4223.351		-1				
4215.766	Zr*	-1				1.006	4223.485		1	2			
4215.813	CN	-2					4223.580		1	1			
4215.976m	Fe-*CN	1	2				4223.733		0	1			
4216.193m	Fe	3d?	5	I	a	0.000	4223.902		-1				
4216.358	Cr V	0	0	III IIA	d	* 0.285	4223.980		-2				
4216.602		1N	1				4224.180m	Fe	4	6	IV	e	
4216.808		-2					4224.305w		0				
4216.903		-2					4224.461		0				
4217.062		-1					4224.818	Fe Cr	3	5	IV V	e d	
4217.207		1					4224.634		-1				
4217.262	-Ru?	1	2			0.808	4224.862		2N	2			
4217.561m	Fe-La*	5d?	6	IV V	d		4225.048		-2				
4217.759		-2					4225.220	V*	0	-1			2.017
4217.880		-2					4225.332	Sa	-2				
4218.056		-1					4225.463m	Fe	3	5	IV	d	
4218.226		1	1				4225.717		1	2			
4218.399		1Nd	1				4225.813		-2				
4218.567		-2					4225.964m	Fe	2	3	IV	b	
4218.736	V-	3N	2	IIIA		0.299	4226.082		-2				
4218.922		-2N					4226.224		-2				
4219.018		-2N					4226.353		0				
4219.199		1	1				4226.433m	Fe	2	2	IV	b	
4219.387	Fe	4	7	IV	b		4226.567		0				
4219.421		3					4226.742m	Ca	20	35	I	a	0.000
4219.597		-2					4226.972		1Nd?				
4219.736		-1					4227.159		-1N				
4219.905		-1					4227.317		1N	-1			
4220.053	V*?	1	1			1.668	4227.442m	Fe	4	5	III	d	

*First Head of Second Band *Cr3.001

FIGURE 2

The curve reproduced in figure 1 was drawn to one-fifth the horizontal and one-half the vertical scale of the original curve, and has been further reduced in the reproduction. Although not suitable for exacting uses, it brings out clearly the progressive deviations. These probably arose from errors in the method of coincidences as used by Rowland and from the adjustments between solar and arc values, and it is their existence that

has greatly impeded efforts to make the wealth of the "Preliminary Table of Solar Spectrum Wave-Lengths" available.

Figure 2 is an illustration of a page of the Mount Wilson revision in the international system of the "Preliminary Table of Solar Spectrum Wave-Lengths." The first column contains the solar wave-lengths in the international system, measured and reduced. Lines too near together for reliable measurement are joined by parentheses. The identifications in the second column have been revised and many new identifications added. The lines due to ionized atoms are indicated by the sign + written as an index to the symbol of the element, and the principal constituent of a multiple identification is underlined. An innovation is introduced in the third column, under disk intensity, by using -1, -2 and -3 for 00, 000 and 0000. This convention gains valuable space and strikingly distinguishes between strong and weak lines.

The four following columns are new and contain data important and convenient to have at hand in physical and astrophysical investigations. The fourth column gives the intensities of the lines in sun-spot spectra relative to their intensities for the disk; the fifth column, the temperature classification from furnace spectra; the sixth column, the classification from the behavior of the lines under pressure, in the explosive spark and in the high-current arc; and the seventh column, the excitation potential of all lines identified in series and multiplets.

The tables are nearly ready for the photographic reproduction and will be published at an early date by the Carnegie Institution of Washington. The revision has been made possible through the assistance and coöperation of Mr. Babcock, Miss Ware, Mr. E. F. Adams and Miss Moore. To their expert and interested coöperation recognition is due in generous measure.

[1] *Recherches sur le Spectre Normal du Soleil*, Upsala, **1868**.

[2] *Trans. Internat. Union for Coöperation in Solar Research, Manchester*, **1**, 1905, p. 157.

[3] *Johns Hopkins Univ. Circular, Baltimore*, No. **106**, 1893, p. 110.

[4] *Phil. Mag., London* (5) **36**, 1893 (49–75).

[5] Bell, Louis, *Astroph. J., Chicago, Ill.*, **15**, 1902 (157–171).

[6] Michelson, A. A., *Ibid.*, **18**, 1903 (278–286); Kayser, H., *Ibid.*, **19**, 1904 (157–161).

[7] Jewell, L. E., *Ibid.*, **21**, 1905 (23–34), p. 27.

[8] Evershed, John, *Observatory, London*, **46**, 1923 (299–304); St. John, C. E., *Mon. Not., R. Astr. Soc.*, **84**, 1923 (95–96); *Mt. Wilson Comm.*, No. **96**; these PROCEEDINGS, *Washington*, **12**, 1926 (65–68); Burns, Kelvin, and Meggers, W. F., *Pubs. Allegheny Observatory, Pittsburgh*, **6**, No. 7, 1925.

[9] *Travaux et Mémoires du Bureau international de Poids et Mesures, Paris*, **11**, 1893.

[10] *Ibid.*, **15**, 1907.

[11] *Trans. Internat. Union for Coöperation in Solar Research*, **2**, 1907, p. 20.

[12] *Trans. Internat. Astr. Union*, **1**, 1922, pp. 35–48.

[13] *Mt. Wilson Contr.*, Nos. **61, 75, 106, 120, 137, 202, 223**; *Astroph. J., Chicago, Ill.*, **36**, 1912 (14–54); **38**, 1913 (341–391); **42**, 1915 (231–282); **44**, 1916 (15–38); **46**, 1917 (138–166); **53**, 1921 (260–299); **55**, 1922 (36–47).

APPLICATION OF THE LAW OF SIMILITUDE TO HYDRAULIC LABORATORY RESEARCH

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Read before the Academy April 25, 1927

Sir Isaac Newton must be looked upon as the originator of the application of the law of similitude to hydraulic laboratory research. In his "Natural Philosophy," published in 1729, we read in Vol. II, under Section 7:

> Suppose two similar systems of bodies consisting of an equal number of particles, and let the correspondent particles be similar and proportional, each in one system to each in the other, and have a like situation among themselves, and the same given ratio of density to each other; and let them begin to move among themselves in proportional times, and with like motions (that is, those in one system among one another, and those in the other among one another). And if the particles that are in the same system do not touch one another, except in the moments of reflection; nor attract, nor repel each other, except with accelerative force, they are as the diameters of the corresponding particles inversely, and the squares of the velocities directly; I say that the particles of those systems will continue to move among themselves with like motions and in proportional times.

While Froude began by applying the laws of similitude to problems of naval architecture, Engels of the Technical University of Dresden, Germany, developed methods for research of constructions of hydraulic engineering.

The proper interpretation of experimental results from models and their transference to the actual is, according to Engels, conditioned not only on the model being made truly analogous but on the flow of water in the model being exactly the same as that in the original; also on the understanding, on our part, under what limitations this analogy exists. To be sure, scientifically exact resemblance between model and original need never be looked for, but in most practical examples one or another motive plays such a striking role as to enable us to speak of analogy. The relations are simplest when we are dealing primarily with acceleration and retardation of water and the causes and effects thereof, and when the action of viscosity of water and of frictional force become less important in comparison (waves, back-water, overflow at weirs, discharge through small orifices, etc.).

From the geometrical analogy between model and construction work are obtained not only mathematical ratios applying to dimensions, but also in connection with the laws of gravity $v = \sqrt{2gh}$ and $h = {}^1/_2gt^2$ legitimate relations affecting velocities, volumes of water, time intervals, forces,

forms of energy, momentums and work. The acceleration (g) in the model cannot be reduced. If in the model

l denotes linear dimensions,	g discharge of water per second,
f surface area,	t time intervals,
w volumes,	p forces,
v velocities,	m momentums,
h velocity heads,	a work,
e energy	

and the same capital letters are used to denote the same items in the construction work; then for a model built at a scale $1:n$ the following relations resulting from the geometrical analogy may be at once written.

$$\frac{l}{L} = \frac{1}{n} \quad \text{or} \quad L = n^1$$

$$\frac{f}{F} = \frac{1}{n^2} \quad \text{or} \quad F = n^2 f$$

$$\frac{w}{W} = \frac{1}{n^3} \quad \text{or} \quad W = n^3 w.$$

For velocities:

$$v = \sqrt{2gh} \quad \text{and} \quad V = \sqrt{2gH}, \quad \text{hence}$$

$$\frac{v}{V} = \frac{\sqrt{h}}{\sqrt{H}}, \quad \text{or since} \quad h = \frac{H}{n},$$

$$\frac{v}{V} = \frac{1}{\sqrt{n}} \quad \text{or} \quad V = n^{0.5} v.$$

For time intervals:

$$t = \sqrt{\frac{2h}{g}} \quad \text{and} \quad T = \sqrt{\frac{2H}{g}} \quad \text{hence}$$

$$\frac{t}{T} = \frac{\sqrt{h}}{\sqrt{H}} \quad \text{or} \quad T = n^{0.5} t.$$

For water discharge per second:

$$q = fv \quad \text{and} \quad Q = FV, \quad \text{consequently,}$$

$$\frac{q}{Q} = \frac{fv}{FV} = \frac{fv}{n^2 f n^{0.5} v} = \frac{1}{n^{2.5}}$$

$$\text{or} \quad Q = n^{2.5} q.$$

For gradient or fall:

$$\frac{i}{I} = 1$$

For the forces since the density γ is constant by taking into account the corresponding masses m and M or the forces p and P

$$g = \frac{p}{m} = \frac{P}{M}, \text{ and since } m = \frac{l^3\gamma}{g} \text{ and } M = \frac{L^3\gamma}{g},$$

therefore,

$$\frac{pg}{l^3\gamma} = \frac{Pg}{L^3\gamma} \quad \text{or}$$

$$\frac{p}{P} = \frac{l^3}{L^3} = \frac{1}{n^3} \quad \text{and} \quad P = pn^3.$$

The quantities of energy per second are

$$e = \gamma qh \quad \text{and} \quad E = \gamma QH;$$

hence,

$$\frac{e}{E} = \frac{qh}{QH}.$$

Since

$$Q = n^{2.5}q \quad \text{and} \quad H = nh,$$

$$\frac{e}{E} = \frac{1}{n^{3.5}} \quad \text{or} \quad E = n^{3.5}e.$$

For momentums:

$$\frac{m}{M} = \frac{1}{n^4} \quad \text{or} \quad M = mn^4.$$

For work:

$$\frac{a}{A} = \frac{1}{n^4} \quad \text{or} \quad A = an^4.$$

The friction of flowing water in flumes and natural water courses should not be disregarded. Let the friction head in the model be r, the friction head in the actual construction be R; then with the designations previously used:

$$v = \sqrt{2g(h-r)} \quad \text{and} \quad V = \sqrt{2g(H-R)},$$

hence

$$\frac{v}{V} = \frac{\sqrt{h-r}}{\sqrt{H-R}}. \quad \text{Now if } \frac{v}{V} = \frac{1}{\sqrt{n}},$$

$$\text{then} \quad h - r = \frac{H-R}{n} = \frac{H}{n} - \frac{R}{n}.$$

Moreover, since $h = H/n$ the condition that must be fulfilled by the conversion formula demands that $r = R/n$; that is to say, the rule for models applies strictly only when the friction head in the model has been reduced in the same ratio as the other linear quantities, including the fall

(or drop) or when the same ratio of friction losses occurs in the model and in actuality.

Whether these conditions are fulfilled by a model may be ascertained with the aid of velocity formula deduced from experiments in flumes of such small hydraulic radii as occur in our models. Biel, for instance, has determined the coefficients of roughness for such flumes, and according to Biel the friction head in a rectilinear canal having a uniform cross-section is

$$r = \frac{ul}{f} v^2 \frac{\psi}{1000}$$

in which r is the friction head in meters, u the wetted perimeter of the canal, l its length, f its cross-section, v the mean velocity of the water $= V/F$ and ψ is a factor which depends chiefly upon the roughness of the channel and on the hydraulic radius f/u. The expression ul/f has no dimensions and is constant for geometrically similar canals and is, therefore, called the "canal constant." If ψ were also constant, then in similar canals

$$\frac{r}{R} = \frac{v^2}{V^2} = \frac{h}{H}.$$

That is, the friction losses would be percentage rise equal for similar canals and the rule for models would be strictly applicable. This is on condition that the friction factors in both model and nature shall be the same; hence, that $\psi = \psi$. But ψ is not constant but varies with the roughness of the channel, with the hydraulic radius and—though in much smaller measure—with the velocity of the water, while the effects of viscosity and density may be disregarded for the temperature variations occurring in actual practice. Biel's experiments have shown that ψ is practically constant for the same roughness, and for large hydraulic radii. It increases only very slowly at first as f/u decreases. But for smaller values of f/u, say commencing with $f/u = 0{,}1\psi$ increases extremely fast. It follows from this that in small scale models, that is, with small f/u, ψ may easily become considerably larger than Ψ. This can be remedied to some extent by polishing the walls of the model as highly as possible.

Reynolds was the first to introduce into such experiments the time element and this meant a long step forward. Operating with a horizontal scale 1:31,800 and a vertical scale 1:960 the time ratios deduced therefrom were

$$\frac{t}{T} = \frac{\sqrt{960}}{31{,}800} = \frac{1}{950}$$

and for scales 1:10,600 and 1:396 the time ratios were

$$\frac{t}{T} = \frac{\sqrt{396}}{10{,}600} = \frac{1}{533}$$

so that during a flood period of 40,700 seconds, the flood period for the first model consequently was found to be 40,700/950 = about 42 seconds and for the second model 40,700/533 = about 80 seconds.

It is a curious fact that the first experiments of hydraulic laboratory research were made on river models and river training which are the most difficult problems of engineering. The difficulties in this kind of hydraulic engineering arise from the fact that we have to deal in these problems not only with the effect of the flow of the water but that in order to obtain results in accordance with what happens in nature the variation of water levels, their slope and discharge, the transportation of debris, the time scale, that is to say, the ratio of a model year to year in nature must be taken into consideration.

Prof. Dr. Krey, of the Prussian National Experimental Institution for Hydraulic Engineering and Shipbuilding, has extended Dr. Engels' investigations in applying the methods of laboratory to a special problem on the river Elbe. It is to be regretted that Dr. Krey's remarkable report on his researches, contained in a large volume, could not be published on account of lack of funds.

I think the most valuable results obtained by Dr. Krey are his finding out, for the special case he examined, the relations between water and debris discharge.

He found out that a certain dependence existed between the quantity of transported debris and the transporting force for which Dubuat has given the formula ti, t being the depth and i the slope.

Dr. Krey found as equation for the quantity of debris

$$M = A(t\,I)^n$$

n being found to lie between 5 and 7. A was ascertained from a long series of experiments for a given depth and constant slope I. As a fairly good formula giving the relation between quantity of debris and transporting force, Dr. Krey found

$$M = \frac{5.3}{10^9}(t\,I)^6,$$

M meaning the quantity of sand transported in cubic cm. per hour and $t\,I$ the force of transportation in grams per square meter. The great difficulty in the evaluation of the quantity of debris carried at different levels lies in the absolute uncertainty as to the depth below the bottom of the river to which the motion of debris extends.

One must realize what the exponent n means—for $n = 5$ to 7 as found by Dr. Krey for the special case he examined it means that with a constant slope i, by doubling the depth, the quantity of debris passing a certain cross-section during the time unit rises to 60- and 100-fold.

SOME FACTORS INFLUENCING THE IGNITION OF CARBON MONOXIDE AND OXYGEN

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Communicated August 12, 1927

In a recent article[2] Professor W. A. Bone has shown that combination can take place between carbon monoxide and oxygen when dried over phosphorus pentoxide for six months, provided sufficient energy is added to the explosive mixture. The condensed discharges used by Professor Bone to ignite the dry mixture were of the order of magnitude of the hot sparks used by Millikan and Bowen in their experiments on stripped atoms, and were amply sufficient to dissociate the gas in the path of the spark into atoms stripped of their valence electrons.

The heavy condensed discharges necessary for the ignition of the dry gas raised the question as to the mechanism involved. The ignition in the absence of water might result from some higher stage of ionization, or it might be that the ignition is independent of any given stage of ionization but is dependent on the introduction of a given amount of energy in a definite volume of gas. The present research was, therefore, undertaken to test this point, as well as to determine the effect of various impurities on the energy of ignition.

The apparatus consisted of an explosion chamber with platinum electrodes placed at right angles to one another. The vertical electrode was placed on a strip of flexible platinum from which was suspended a piece of iron enclosed in glass; it was thus possible to vary the spark gap by means of an external electro-magnet.

The condensed discharge was obtained from twelve 1500-volt 1-mf. condensers which could be placed in parallel and charged to any required potential by means of a high voltage d. c. generator placed across a suitable potentiometer.

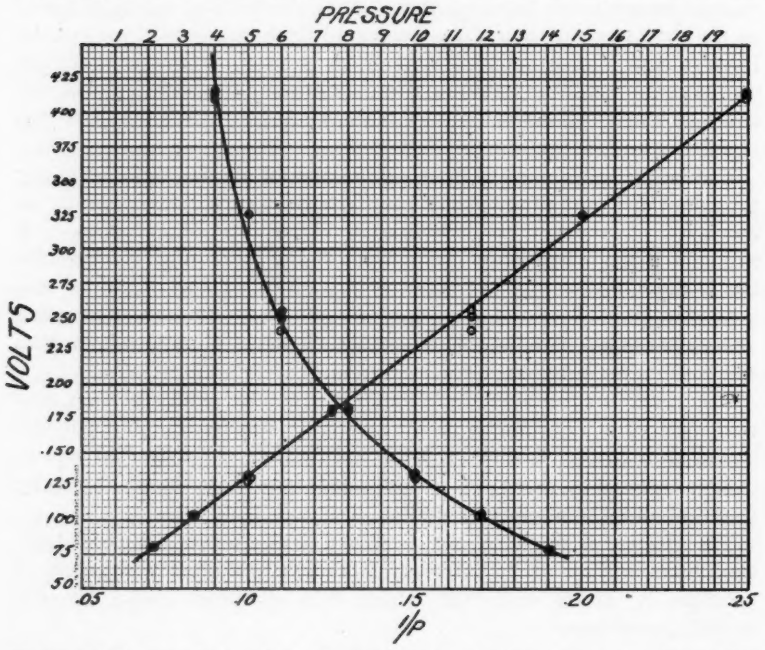
The explosion chamber was connected through a ground glass stopcock to a reservoir train where an explosive mixture of carbon monoxide and oxygen could be generated and stored up either over water or sulfuric acid. Another gas train was provided so that the vapor of any liquid could be introduced at any given pressure up to the vapor pressure of the liquid. The pressures were read by means of mercury manometers. A Hyvac pump was used for evacuation.

All stopcocks were lubricated with high vacuum stopcock grease, as it was shown by experiment to have no detectable effect on the ignition of the gases in question.

The mechanism of ignition of the explosive mixture of gas at a constant

pressure was shown to be independent of any particular stage of ionization produced in the spark by placing inductance in the discharge circuit. A small inductance capable of cutting the peak current density to about one-tenth its normal value as well as a large inductance which reduced the current density much farther had no appreciable effect on the ignition point of the gas over that obtained with a similar resistance in the circuit but without the inductance. This was true for both the moist gas and the gas dried by sulfuric acid at 14 cm. pressure.

The ignition at constant pressure was shown to be dependent on the



energy of the spark in which the voltage for ignition was determined for different condenser capacities. The energy of the spark is given by $E = {}^1/_2CV^2$.

The results obtained are listed in table 1.

TABLE 1

CAPACITY	VOLTAGE	ENERGY × 10^6 JOULES
12 mf.	290	504,000
11	310	528,000
9	344	532,000
6	437	570,000
3	616	570,000
2	785	617,000

It will be seen that the energy necessary to ignite a given pressure of gas varies but little for different voltages, increasing slightly for increasing voltage. This slight increase in energy is readily accounted for by the greater distance the spark jumps at the higher voltages, thus distributing the energy over a slightly larger volume.

The effect of the pressure of the explosive mixture on the voltage necessary for ignition is shown by the accompanying curve.

In these experiments the capacity was kept constant at 11 mf. The experiment was performed with moist gas.

It will be seen that the pressure-voltage curve is a hyperbola where $V = k/P$, where P is the pressure of the explosive mixture.

From these data it will be seen that while the ignition at any one pressure is determined by the energy of the spark and not by the voltage, the ignition points at various pressures are related not as the energy ($^1/_2CV^2$) but as the voltage (V).

The effect of water vapor on the energy of ignition is shown by the following data.

TABLE 2

PARTIAL PRESSURE	IGNITION VOLTAGE
0.4 mm.	262
1.0	182
1.6	120
2.0	101
4.0	71
6.0	60
10.0	54
20.0	46

In these experiments the pressure of the explosive mixture dried by sulphuric acid was maintained at 14 cm. and the pressure of the water vapor added varied.

The combustion of the explosive mixture dried by sulphuric acid seldom went to over 10% completion; likewise the flame lacked sufficient energy to propagate itself throughout the entire explosion chamber. Only the most minute quantities of water vapor were necessary to cause the reaction to go to completion. Long-continued exhaustion followed by many explosions in the reaction chamber was necessary to remove the water vapor sufficient for incomplete combustion.

Unfortunately adsorption on the walls of the explosion chamber made it impossible to determine accurately the pressure of a vapor such as water to 0.1 mm. The per cent experimental error is, therefore, high for the very low pressures.

The impurities other than water which were added to the explosive gas may be divided into two general classes, those which are inert in the explosion except in that they absorb energy, thus increasing the ignition

potential, and those which in themselves are oxidized in the explosion and thus lower the ignition potential.

In the following table are listed the various impurities which, when added in appreciable quantities, increase the energy of ignition. In the table are given the partial pressures for the different gases which, when added to 14 cm. of explosive mixture (2 cm. water and 12 cm. $2CO + O_2$), increase the voltage of ignition from 100 volts to 400 volts, or increase the energy of ignition 16-fold.

TABLE 3

GAS	PARTIAL PRESSURE	GAS	PARTIAL PRESSURE
Argon	26 cm.	Alcohol	3.8
Nitrogen	20	Isopropyl nitrite	0.7
Carbon dioxide	9.4	Ether	0.66
Chloroform	4.2		

It will be noted that the effect of the added impurity on raising the voltage of ignition increases with the increasing complexity of the molecule added. Unfortunately, the energy going into dissociation and the specific heat of the gas under ignition conditions cannot be told accurately. Rough calculations, however, indicate that the ability of a gas to raise the ignition voltage depends directly on the amount of energy that it absorbs.

The second class of impurities which ignite during the explosion affect the voltage of ignition in a manner identical to that of water, i.e., greatly reduce the ignition voltage, and cause the oxidation to go to completion when present even in traces. In order that these results might be obtained it was essential to add oxygen sufficient to burn completely the ignitible impurity.

The pure explosive mixture dried by sulphuric acid ignited from between 375 and 400 volts for an 11 mf. condenser capacity, varying somewhat from time to time. In the following table are listed the partial pressures of ignitible impurity which when added to 14 cm. of explosive mixture reduced the energy of ignition to 200 volts, to 100 volts and to 50 volts.

TABLE 4

VAPOR	PARTIAL PRESSURE 200 *V*	100 *V*	50 *V*
Water	1.0 mm.	2.0 mm.	20.0 mm.
Hydrogen	0.4	1.2	12.0
Alcohol	0.6	1.8	10.0
Ether	0.2	1.0	6.0
Carbon disulphide	3.2	9.0	18.0

It will be noted that a vapor, such as alcohol or ether, either raises or lowers the ignition voltage depending on the amount of oxygen present. For instance, pure ether vapor, when added to the moist explosive mixture at 14 cm. pressure, lowered the ignition point up to a concentration of 0.7%,

beyond which the ignition voltage rose rapidly. When, however, sufficient oxygen was added to burn the ether a pressure-voltage curve similar to that given for water was obtained.

The partial pressure-voltage curve for the added carbon disulphide was a hyperbola giving $V = k/p$, where p refers to the pressure of the added disulphide. This curve was identical in form to the pressure-voltage curve obtained in the ignition of the $2CO + O_2$ mixture where P referred to the total pressure. From this relationship it may be concluded that the voltage necessary to ignite a constant volume of explosive mixture in the presence of carbon disulphide is inversely proportional to the number of molecules of disulphide present.

The partial pressure-voltage curve for water, hydrogen, alcohol and ether were distinctly different from that for carbon disulphide in that the point of inflection beyond which additional vapor had but little effect on lowering the voltage was much sharper. While the per cent experimental error was necessarily very large in measuring the lowest pressures of the added vapor, the curves were nevertheless fairly well expressed by an equation of the form $V = k/pe^{\beta p}$, where p refers again to the partial pressure. It is interesting to note that the value of β came out to be 0.10 in every instance.

The fact that the partial pressure-voltage curves for hydrogen, alcohol and ether gave values of β practically identical to that obtained for water, combined with the fact that these gases acted as positive catalysts only when in the presence of sufficient oxygen for complete combustion, suggests that their positive catalytic properties are derived largely, if not entirely, from their ability to furnish water in the explosive gas. If this contention is correct then it follows that, since these vapors are more efficient in lowering the ignition voltage than is water, the water produced from the burning vapor (new-born water) is a more potent catalyst for the oxidation of carbon monoxide than is ordinary water.

The catalytic properties of carbon disulphide may be explained in a similar manner. Sulphur dioxide in small quantities showed itself to be a very weak positive catalyst; however, it is very possible that the "new-born" sulphur dioxide from the oxidized carbon disulphide possesses stronger catalytic properties.

It might be suggested that the combustible vapors lower the ignition voltage for the explosive mixture by raising the temperature as a result of their own combustion. Such a simple picture hardly seems probable, however, since the energy derived from the oxidation of the vapor is extremely small compared with the energy of the igniting hot spark discharge,

A further series of experiments was performed in the hopes of throwing light upon this point. The conditions were maintained identical to those in which the data in table 4 were obtained except that the carbon mon-

oxide of the explosive mixture was replaced by nitrogen. Comparative data are given in table 5.

TABLE 5

VAPOR	PARTIAL PRESSURE 800 V	200 V	100 V
Hydrogen	4.0 mm.	11.0 mm.	18.5 mm.
Alcohol	6.0	20.0	
Ether	4.4	8.0	20.0
Carbon disulfide	2.4	23.0	

A comparison of the data in tables 4 and 5 reveals the fact that the amount of vapor necessary to bring about the ignition of the carbon monoxide-oxygen explosive mixture is many times less than an amount which can undergo ignition in the absence of the carbon monoxide. From this it is possible to draw the very interesting conclusion that two explosive gases in an explosive mixture may ignite more readily than will either gas separately under similar conditions.

The pressure-voltage curves for the vapors listed above were all hyperbolas similar to that obtained from the ignition of the pure carbon monoxide and oxygen. Thus the ignition voltage will be seen to be inversely proportional to the pressure, hence the number of molecules in the path of the spark. In this respect it will be seen that the present curves differ materially from the partial pressure-voltage curves obtained when these vapors were added to the carbon monoxide-oxygen explosive mixture. It, therefore, seems very improbable that the lowering of the ignition voltage of the latter by the presence of these vapors can be due to the energy liberated by their ignition.

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[2] Bone, *Proc. Roy. Soc.*, **110**, 615 (1926).